RELATIONSHIPS BETWEEN SUBSET AND ONE-POINT SOLUTIONS IN COOPERATIVE GAME THEORY: A WORKER COMPENSATION APPLICATION

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ABSTRACT

Cooperative games are most frequently used to divide results among members who join in a cooperative situation. We've done a thorough analysis of the subset and one-point solutions as a result of the wide range of options provided by this method. In our analysis, we especially considered four subset solutions and six one- point solutions and examined their relationships. The large number of solutions we have taken for investigation at once is what distinguishes this study. We have also compared and discussed these solutions to a problem that has to do with rewarding workers. The analysis of this topic is different since there are four players instead of only two or three as in previous research. In order to make the obtained results as clear as possible, we have also constructed a schematic representation of the problem's solutions. We conclude that in order to allocate the results fairly, a thorough analysis of as many potential solutions as feasible should be made before a compromise is reached between the parties.

Keywords: Cooperative game, Subset solution, One-point solution, The core, The Nucleolus, The Shapley Value.

INTRODUCTION

Albanian emigration has increased dramatically in recent years, which is extremely concerning. According to INSTAT statistics for 2021, there are 1.68 million Albanians left around the world (Institute of Statistics, n.d.). The majority of those who have emigrated in recent years have been doctors and nurses who have seen their salaries rise as a result of the pandemic. Lack of workers in Albanian agriculture is another concerning issue. Instead of helping their place of origin, many decided to labor in foreign nations. This is also a result of the increased salaries and greater labor expenses in the emigration nations. In this form, a very large collapse has been caused about by a lack of laborers in the construction, tourism, and agricultural industries. This is also reflected also by a study from Euopean Training Foundation ("How Migration, Human Capital and the Labour Market Interact in Albania | ETF," 2021). Due to a severe lack of workers, those volunteers who still want to work are more forceful and claim to be the only ones who can find employment. The workers demand from the Albanian owners a significantly larger salary than they did previously. On the other side, business owners in various industries also experience pressure from the workforce and the desire to maximize their profit. In this way, we are dealing with a conflict between the parties' subjectivist interests. According to a brief investigation of Albanian farmers, workers were paid according to the number of hours or days of work that were estimated for each unit of time. In many other situations, the owner and the workforce have reached preliminary agreements that rely the payment on the amount of produce at the end of the season. So, we are in situations in which decision making is shared among several parties (owner or employee). As a result, there is a need for analytical modeling of these situations, which is based on appropriate quantitative analysis and principles that are accepted by all parties. Many times, we are aware that we should receive less (more) benefit in a circumstance, but the exact amount is a major issue that is the subject of much debate. Non-cooperative games (NCG) and cooperative games (CG) both theories are focused on specific elements of player competition and cooperation. Cooperative does not mean that all agents are agreeable and obey arbitrary orders. It implies that groups, not individuals, are the fundamental modeling unit. In the NCG players compete against one another, pursuing their own selfish interests in order to maximize their own profit. However, in the CG players work together by joining up in groups, so-called coalition. In the NCG nobody gets together because everyone is fighting for themselves. In CG players work together to achieve their aims and gain from working in coalitions if doing so increases their individual profit. Since a NCG can be understood as a CG whose coalitions are singletons, cooperative games may be considered the more general concept. Though there are several techniques to generalize NCG to CG.

Cooperative games can be with non-transferable utility (NTU) and with transferable utility (TU). A cooperative game with TU means that the value can be distributed arbitrarily between members of a group. Because in the non-cooperative game with NTU, the reward that a user receives in a coalition is fixed, its value cannot be measured by a function. Any two players in CG with TU can compare their utility, and the utility can be shared among agents. A cooperative game with transferable utility (TU game) is given by a pair $G = (N, v)$, with the set $N = 1,2,\dots,n\}$ of players and the characteristic function also referred to as the coalitional function $v : 2^n \to \mathbb{R}^+$. Cooperative games are also called the coalitional games. A coalition may represent a number of individuals or a group of individuals (labor unions, communities). Each player's value for standing alone during the game is denoted by the letter $v(i)$. Each subset (or coalition) $S \subseteq N$ of players indicates the utility (or gain) $v(S)$ that they attain by working together. The value of creating the grand coalition, which consists of all users, is given by v(N). It is also vital to decide how these gains will then be distributed among them. Commonly we assume that the characteristic function v of a TU game $G = (N, v)$, satisfies the normalization and monotonicity properties. When a game is monotonic, a larger coalition has a higher value than a smaller coalition, and when a game is normalized, the value of a null subset is equal to zero. The two main issues in CG are what kind of coalition would be possible to form and how the benefits will be distributed among the participants. A payoff distribution vector $x \in \mathbb{R}^n$ that offers a value to each player in the coalition serves as the game's solution. Different players (coalition members) have different ideas about how to divide the common profit. A distribution must be stable in order for all of the members to accept it. When a coalition is stable, no one wishes they were in another coalition or on their own. There could be several stable sets for a problem, which is the biggest problem with the stability idea.

In game theory, CG can be used for both cost allocation and profit distribution. The goal of the cost allocation game is to maximize cost savings as much as possible. A group of agents who are willing to work together on a project are present in a cost allocation problem. The project has some components that all agents must use, but it also has some unique ingredients that are not found in other projects. Allocating the costs resulting from the collaborative performance of the project that includes all of the particular features defined by the agents is the greatest difficulty in a cost allocation problem. A cost allocation problem can be modeled as a benefit allocation problem and vice versa.

Regarding the issue with the employment of employed workers that we'll explore in the practical application. There aren't any actual studies examining the employment of seasonal workers in various industries in Albania. The use of descriptive and quantitative analysis to this issue is still open. By carefully examining the cooperative game theory, this issue has been clarified in the work (Kedhi & Bekolli, 2022). Three distinct circumstances that generally involve the hiring of human workers have been explored in the work of (Kedhi & Bekolli, 2022). The owner and two workers are present in the first two scenarios, whereas the third scenario features more workers thanks to an analysis of games with a coalition structure. Our work is interesting because, in addition to Shapley value and nucleolus, we consider other solution concepts.

LITERATURE REVIEW

A.Cooperative Game Theory

The famous book by Von Neumann and Morgenstern contains the first mention of the cooperative game theory (Neumann & Morgenstern, 1944) . In a cooperative game, the players cooperate to accomplish their goals, claims Fragnelli (Fragnelli, 2010). These goals may be the same for all players or they may differ. The primary presumption is that players are rational and always consider how others are acting. Many different cost-sharing situations have been modeled using cooperative game theory. Generally, the theory of cooperative games investigates and supports those circumstances whose solutions involve all player in the game. The Tennessee Valley problem was one of the first programs designed to distribute costs (Parker, 1943). Since then, it has become clear that there is no technique for cost distribution that is hundret percent effective. Parker's research opened the door for the implementation of cooperative games in terms of cost allocation. Although modeling circumstances that attempt to distribute profits has a structure quite similar to that of cost, applications of this kind are less common in the literature. Ponte (Ponte et al., 2016) claim that cooperative game theories offer a useful analysis for supply chains and the distribution of the net benefit obtained from cooperation. In the work of (Frisk et al.,2010) and in the area of technology, certain applications for cost distribution utilizing the theory of cooperative games can be suggested (Saad et al., 2009). The water resources concept has a wide variety of uses for cooperative games. The cost is divided among the water users in the study by using cooperative games (Deidda et al., 2009).Additional applications in the development of water resources include (Do, 2019),(Lippai et al.,2000), and (Sechi et al.,2012).In addition to the issues we have with water, runways are also used for cooperative games (Littlechild, 1974). The book by Young (Young, 1994) offers a wide range of applications. The energy industry has seen a significant number of applications that have utilized cooperative game concepts (Tan & Lie, 2002). The book by Young (Young, 1994) offers a wide range of applications. Two concepts, the *player set* and the *characteristic function* of the game, are crucial for the modeling of cooperative games. Game players don't always have to be human individuals; they can be stand in for organizations, brands, or companies. Furthermore, the study of the formation of the coalition and the distribution of the profit at the end of the game is done based on the values obtained by the characteristic function. According to Fragnelli (Fragnelli, 2010), there are two primary subgroups of solutions to issues that are described as cooperative games with transferrable utility: subset solutions and point solutions. Numerous such have been taken into account for the solution of cooperative games. This article's main goals are to first illustrate the relationship between these solutions for cooperative games and, second, to illustrate it with a numerical application. The imputation set, the game's core, the weber set, and the core cover are a few possible subset solutions. In addition, we examined the Gately value, modiclus, nucleolus, pre-nucleolus, Shapley value, and τ-value in the one-point solution.

B. Solutions Concept

In mathematics, the solution to a cooperative game is a function based on criteria, axioms, and principles, not just a single formula that divides the outcome among the players. Each solution provided by cooperative game theory is chosen from a set known as the set of imputations. The reason it is chosen by this set is that it fulfills two very important properties for maintaining the stability of each member in each coalition.

Definition 1. Let be a coalitional game $G(N, v)$ with TU and x a payoff vector from $\mathbb{R}^n$. This payoff vector is called an imputation if it satisfies the following conditions:

1. *Payoff vector x is individually rational, so every player benefits more than they would by remaining alone , i.e $x_i \geq v(i)$*
2. *Payoff vector x is efficient , so all the value $v(N)$ is distributed among the members of the grand coalition, i.e $\sum_{i=1}^{n} x_i = v(N)$*

The set of all imputations of the game $G(N, v)$ is denoted by $I(G)$ and, in a mathematical way, is written as follows: $I(G) = \{x = (x_1, x_2 \dots x_n) : \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v\{i\}, \forall \in N\}$

The largest set from which further subsets will be needed to produce the game's "solution" is the imputation set. The pre-imputation set is the collection of all the profit vectors that satisfy just the second criterion and not the first, and we denote it symbolically with $I^0(G)$. We are intrigued to games where the extension of a coalition results in a greater reward than the original coalition. The players are encouraged to create the grand alliance in this way. We will assume that the players will form the grand coalition

Definition 2. A function $\mathcal{F}$ which assign to every cooperative game $G(N, v)$ a possibly subset $\mathcal{F}(v)$ of $\mathbb{R}^n$ is called a solution concept.

A distribution must be stable in order for all of the members to accept it. When a coalition is stable, no one wishes they were in another coalition or on their own. There could be several stable sets for a problem, which is the biggest problem with the stability idea. There is a set in which no players choose to deviate in order to create other coalitions, the "core of a game".

Definition 3. The core of a TU coalitional game $G(N, v)$ is the set of all payoff vectors $x \in \mathbb{R}^n$ that satisfy the efficient, individually rational, and coalitional rational properties. So, the core of $G(N, v)$ is the set: $C(N, v) = \{x \in \mathbb{R}^n | \sum_{i \in N} x_i = v(N) \text{ and } \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N\}$

The fair and stable payoff vectors are all included in a game's core. All of the core components are agreed accepted by all of the game's players. A game's core can be either infinitely set or empty set. When the core is empty, it indicates that there isn't any allocation that all of the members can accept to. It has been proven that the game's core is not empty in balanced games.This was the most important result proved by Bondareva and Shapley.

The core of the game, first suggested by Gillies, is one of the most significant subset solutions for cooperative games (Gillies, 1959). Although according to other authors (Weiss & Shubik, 1984), the game's core was developed by Shapley as a concept for a solution. It is interesting to study the games that have non-empty cores. All of the large coalition's members receive a more fair and stable allocation as a result. For this, a sufficient condition for the game's core to be different from empty has been demonstrated; it is sufficient for it to be a convex game (Shapley, 1971). Due to the fact that it satisfies three of the requirements for being a viable solution, the game's core is regarded as one of the most significant solutions in cooperative games. The core satisfies efficiency, individual and group rationality. Due to the cooperative game's structure, it often turns out that the core is empty; however, in a cooperative game, an empty core indicates that no allocation can be made that is stable and accepted by all players. An approximation of the core with epsilon has been presented for these cases.

Nucleolus and modiclus are two solution concepts that are strongly related to the game's core. The nucleolus of a cooperative game measures the degree of satisfaction of each coalition. Modiclus treats all coalitions as equally as it can, in contrast to the cooperative game nucleolus. Modiclus in much literature is known as "modified nucleolus". Modiclus treats all coalitions in a cooperative game on equal terms (Sudhölter, 1996).

The formula $d(S,x) = v(S) - \sum_{i\in S} x_i$ determines the surplus of the coalition S in respect to a profit vector x from $\mathbb{R}^n$. Comparing two imputations of a game using the lexicographical technique is a notion that is introduced in order to define the nucleolus of a game. Let's mark with $\Theta^v(x)$ the vector which has as coordinates all surpluses $d(S,x)$ of each coalition S written in descending order.

<u>*Definition 4.*</u> *Let x and y be two elements from the set $\mathbb{R}^n$. The vector x is lexicographically smaller than or equal to y if either $x = y$, or $x \neq y$ and for $i = min\,\{j \in \{1,2 \ldots k\}: x_j = y_j\}$ we have that $x_i > y_i$. Symbolically, we write it by $x \preccurlyeq_{lex} y$.*

According to a lexicographical comparison, all of the profit vectors in the core are such that there are no other vectors that are superior to them. A vector x is said to be the nucleolus solution if it prevents any coalition S from growing one surplus without decreasing another. We have the following definition in a more formal way:

<u>*Definition 5.*</u> *The nucleolus of the cooperative game $G(N,v)$ is called value:*

$$\boldsymbol{N(G)} = \{x \in I(G): \Theta^v(x) \preccurlyeq_{lex} \Theta^v(y), \forall\, y \in I(G)\}$$

The definition of the *prenucleus* of the game is obtained if the imputation set in the previous definition is swapped out for the preimputation set, making $I(G)$equal to $I^0(G)$.

If the vector of surplus differences sorted in a non-decreasing way is used to obtain the modiclus, then the vector of surplus differences ordered in the set of the optimal payoff vector is used to obtain the nucleolus. It has been proven that exist the nucleolus of a coalitional game G with utility transferability for all non-empty and compact subset $X \subseteq \mathbb{R}^n$. When the game's core isn't empty, the nucleolus and prenucleus are there, but for modiclus, this isn't always the case. The key advantages of the nucleolus and the prenucleolus, if players choose to use this notion for their answer, are that they are a single point and do not encourage any discussion. Their computation, for which many researchers have investigated, is, however, its main drawback. The majority of the time, linear programming is used to calculate them analytically and iteratively. Even though it's not thought of as one of the easiest, as the number of participants rises. The existence of a nucleolus or a prenucleolus in a payoff vector is frequently determined by criteria. Kohlberg's standard can be mentioned. Both of these concepts, nucleolus and prenucleolus, provide the same profit vector if the game's core is not empty.

Schmeidler was the first to put up the idea of nucleolus (Schmeidler, 1969). In his work, he demonstrated that the nucleolus of a cooperative game is always present and consists of a single point. Schmeidler also demonstrated that when the game's core is not empty, the nucleolus is in it. The nucleolus and its characteristics have been the subject of extensive study. It is axiomatized in further detail in Potters (Potters, 1991) and Snijders (Snijders, 1995). The cooperative game's nucleus' computation is one of its main weaknesses. In order to make the calculation of it relatively simple to find the game's nucleolus, several methods have been developed. We can name a few of them: (Sankaran, 1991), (Dragan, 1981), (Puerto& Perea, 2013), and (Faigle et al., 2001). The Kopelowitz algorithm is one of the more popular ones since it utilizes the complex linear programming method multiple times to find a solution (Kopelowitz, 1967). In a recent study, the nucleolus of the game was examined to determine how energy losses were distributed (Songhuai et al., 2006).

Another subset solution of cooperative games is the one called Kernel (Davis & Maschler, 1965), (Maschler & Peleg, 1967). In cooperative games, the kernel is the subset of the core when it exists, as demonstrated by Granot and Driessen (Granot, 1995; Driessen, 1998).

Despite the fact that the game's core offers a wide variety of answers, players frequently seek for particular ones. A notion that is very useful in cooperative games is the marginal contribution of the players. The marginal contribution of a player i in a coalition tells us how much he is increasing the value of that coalition after the player's i joining. We indicated it with $m_i = v(S \cup \{i\}) - v(S)$ for every coalition $S \subseteq N.$ The marginal contribution of a player depends on the choice of permutation in which he participates. The author of the book by the same name initially proposed the Shapley value (Shapley, 1953). In the works of (Pintér ,2015), (Chun ,1989), (de Clippel,2018), and, a more thorough axiomatization of this value may be found (Skibski, 2014). The work of (KAMIJO, 2009), which presents a simpler approach for computing the Shapley value based on cooperative games with coalition structures, is worth mentioning as one of the most recent scientific advances related to the solutions of cooperative games. Using the Shapley value recently has yielded several worthwhile results, including those of the authors (Chen & Tang, 2017). Shapley's value is one of the most important solutions for cooperative games. All coalition permutations are taken into account while calculating the Shapley value, which is a point solution for cooperative games. In a cooperative game with n players, it is calculated as the average of n! marginal vectors.

Definition 6. The Shapley value of player i *of a cooperative game* $G(N, v)$ *is denoted by* $\Phi(N, G)$. *It's* $i-th$ *cordinate is given by*

$$\varphi_i(N, v) = \frac{1}{n!}\left[\sum_{S\subseteq N\setminus i} |S|!\,(n-1-|S|)!\,[v(S\cup\{i\} - v(S)]\right] = \frac{1}{n!}\sum_{\sigma\in\Pi} m_v^{\sigma}(i)$$

In which S is a coalition that does not contain player i. The more players there are, the more complicated the computations are to determine the Shapley value. The Shapley value satisfies a number of desirable axioms. The sharing of all the benefits of the large coalition is guaranteed by the efficiency axiom. According to Shapley, the axiom of symmetry states that participants who make the same marginal contribution will obtain the same distribution. According to the axiom of the dummy player, the player who makes the same marginal contribution to each coalition is rewarded with the value he would have earned if he played alone. Finally, the Shapley value establishes the additivity axiom, which states that splitting a game into two other parts does not result in better outcomes. The only point solution that satisfies these logical requirements is the Shapley value. Even when the game core is empty, the Shapley value always exists. Furthermore, the Shapley value might or might not be at the game's core. In the case of super-additive games, the Shapley value is an imputation, so it satisfies the requirement of individual rationality.

As for Weber set, it is a set-valued extension of the Shapley value. Weber set is a subset solution for which it has been demonstrated that the game's core is always present there, particularly when the game is convex, the weber set and the core coincide (Curiel, 1997).

Definition 7. The Weber set of a coalitional game $G(N, v)$ is the convex hull of its marginal vectors: $\mathcal{W}(N, v) = conv\{m^{\sigma}(v) : \sigma \in \Pi(N)\}$

Given that every marginal vector is an efficient vector, we can conclude that the weber set of a cooperative game is always a non-empty set that is also convex and compact. Weber showed in 1971 that the core set of each coalitional game is a subset of the Weber set so, $C(N, v) \subseteq \mathcal{W}(N, v)$. Every game has a Weber set, and convex games have the core set that is identical to the Weber set.

Weber set and another solution called Tau value are two concepts that are strongly related to the upper and lower vector. These vectors show the minimum and maximum values that players can achieve in the game. The marginal contributions of the players in the grand coalition define the coordinates of the upper vector $b = (b_1, b_2 \dots b_n)$, so $b_i = v(N) - v(N - \{i\})$. Additionally, the equation $a_i = max_{S \subseteq N : i \in S}\{v(S) - \sum_{j \in S - \{i\}} b_j\}$ provides the coordinates for the lower vector $a = (a_1, a_2, \dots a_n)$. Only for quasi-balanced games is the τ −value in comparison to the Weber set determined. It is important to note that a cooperative game never has an empty Weber set. This solution is very important considering that we mentioned above that some very important subset solutions, such as the core, may not contain any elements.

The idea of Tau value was first presented by Tijs (Tijs, 1981) as a concept for cooperative game solutions. More details about the axiomatization of tau value are available in (Tijs ,1987). Tau value has the disadvantage of not always being found in the core of the game, like other one-point solutions. Driessen and Tijs (Driessen & Tijs, 1985) set conditions for which tau value is at the core of the game

Definition 8. The τ −value for a quasi-balanced cooperative game $G(N, v)$ is $\tau = a + \lambda b$. The value λ is chosen in a manner that $\lambda = \begin{cases} 0 & if\ a = b \\ \frac{v(N) - \sum_{i \in N} a_i}{\sum_{i \in N} b_i - \sum_{i \in N} a_i} & otherwise \end{cases}$

Even when the game's core has non-empty cores or is convex, the τ −value is not always located there. However, it satisfied the axiom of dummy player and symmetry.
The core cover of a cooperative game is the final subset solution that will be discussed in the case study.

Definition 9. The core cover of a coalitional game $G(N,v)$ is the set of all imputations that lie between the upper and lower vectors. Symbolically, the covering of the core is written as follows: $CC(v) = \{x \in I(G): a \leq x \leq b\}$
When a cooperative game has non empty core, Tijs and Lipperts in 1982 demonstrated that its core is a subset of its core cover.

To reduce the temptation for players to leave the major coalition as much as possible, Gately Point is utilized as a tool. The same-named author developed the concept of the Gately point (Gately, 1974). This idea tries to reduce participants' overall propensity to desert from the grand coalition. This technique was initially tested in a profit distribution issue with just three players. Then it is expanded to the situation with a total of n players in Littlechild and Vaidya (Littlechild & Vaidya, 1976). The Gately point is discussed in more depth and research in (Dhamal et al., 2019). To determine the Gately point, first determine each player's propensity to disrupt for payoff vector x inside the imputations set for all players i from N using the formula $d(i,x) = \frac{v(N)-v(N-\{i\})-x_i}{x_i-v(i)} = \frac{M_i-x_i}{x_i-v(i)}$. This shows the split that would result if the player left the grand coalition. The goal is to identify an imputation that has the least potential to ruin the grand coalition. The values of the coalitions of size 1, n-1, and n are the only factors that affect this solution concept.
For superadditive games, when the value of two coalitions is higher than or equal to the sum of their individual values, the formation of a grand coalition is justified. Clearly, the big coaliton provides the most rewards. The following description of game superadditivity provides the formal definition.

Definition 10. A cooperative game $G(N,v)$ is said to be superadditive if for all $S,T \subseteq N$ such that $S \cap T = \emptyset$ there is true $v(S) + v(T) \leq v(S \cup T)$.
One of the other most important properties that a game can have is that it is convex. This property is both similar and different from that of superadditivity. In both cases, when a game fulfills these properties, it is in the interest of every player to create the big coalition.Due to the fact that it considers coalitions with common elements, convexity is a stronger criterion than superaditivity.

Definition 11. A cooperative game $G(N,v)$ is said to be convex if for all $S,T \subseteq N$ there is true $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$
The convexity of a coalitional game $G(N,v)$ implies superadditivity, but the opposite isn't always true, as you can see in the case study. There are several games that satisfy the superadditivity condition, but they are not convex. The convex game has many advantageous characteristics, however this isn't the objective of this paper. We'll analyze a specific game that doesn't fulfill this property.

Another class which is also very important is the class of all balanced games. The case study that will be analyzed is a balanced game. However, in order to explain this notion, we must first define balanced maps.

Definition 12. *Let be the set* $N = \{1,2 \dots n\}$. *A map* $\lambda: 2^N \setminus \{\emptyset\} \to \mathbb{R}^+$ *is called a balanced map if* $\sum_{S\subseteq N} \lambda(S)e^S = e^N$ *, where* e^S *is a characteristic vector or coalition* S *such that*

$$e_i^S = \begin{cases} 1 & if \quad i \in S \\ 0 & if \quad i \in N\setminus S \end{cases}$$

Definition 13. *A coalitional game* $G(N, v)$ *is called a balanced game if for each balanced map* λ *there is true that* $\sum_{S\subseteq N} \lambda(S)v(N) \leq v(N)$

A game being balanced could be interpreted as meaning that having a grand coalition operate for one unit of time is at least as productive as having a balanced distribution of time over several smaller coalitions, with the worth of coalitions being interpreted as productivities. Therefore, it would seem advantageous to form the grand coalition in a balanced game. Additionally, it is advantageous for a big coalition to form in a game when it produces a better outcome than the sum of the values of all individuals acting independently. These types of games are called "essential games."

METHODOLOGY

Our paper seeks to examine some of the fundamental characteristics of cooperative game theory, as well as the solutions that this model provides. This approach is distinct and interesting since we also took into account several solution ideas that are uncommon in the literature, like the Gately value or the Weber set. In the theoretical part, we've briefly and briefly illustrated the relationship between some solution concepts and the circumstances in which they don't even exist. Without knowing the precise values of the solutions, we may still decide which of them is best and most agreeable to all coalition members if we have enough knowledge of the theoretical components of a cooperative game. To avoid limiting the paper to the boundaries of the theoretical presentation, we selected for a case study. The chosen case study is quite practical, and by using it, we hope to provide an illustration of how the cooperative game theory model may be used in real contexts. To make the application beneficial for resolving issues between the owners of Albania's agricultural industry and the workers who are always wanting to leave the nation in particular, we have related this case study with the issues we outlined in the introduction of this paper. The value of the characteristic function obtained in the study is an all-inclusive value by applying the percentage profit sharing among the members. Regarding the range of ages available for employment and the output they produce, these factors will vary in practice and may have various values, but the process for choosing participants, the defining characteristics, and the underlying presumptions may not change. We have collected the results after deciding which components the model we developed for the case study will include. Given the game's characteristics, we have determined how to examine the results from both a theoretical and a numerical perspective. we calculated the numerical results with R package, and precisely with coopgame. The same methodology can be used even if a higher number of players or a different value for the characteristic function are used in another case study.

A NUMERICAL APPLICATION OF EMPLOYEE REWARD (Number close to reality)

An owner looks to hire staff during the product's harvest season (specifically, the cherry harvest in the July season in the county of Korca). The owner must hire at least two employees due to the nature of the product. The owner will be more profitable the more staff he has since he will be able to utilize the complete output. Four workers simultaneously show up for the owner's notice. These people range in age from 18 to 38 years old. The personnel on display have varying levels of experience product gathering experience. Employing every employee is in the owner's best interest because doing so allows him to utilize the farm's full potential even more quickly. The following findings are based on the employees' prior experiences and the basis of each individual's yield over a period of one month or as long as the season continues:

If just the first and second employees are hired, they may each gather only 15% of the farm's output in a month; the first and third workers together collect 20% of the product; the first and fourth collect 17%; the second and the third collect 30%; the second and the fourth collect 25%; the third and the fourth collect 35%; the first, second, and third employees together gather 40% of the farm's output in a month; the first, second, and fourth collect 50%; the first, third and fourth collect 40%; the second, third and fourth collect 65% and all the fourth together collect 100% of the output.

The owner claims to have an amount of money ready to pay the employees for the entire harvest of his production. We'll presume that the employee will give 100 units of payment. The exact percentage distribution of the outcome among the workers will not matter because we shall do it in this manner regardless of the outcome. We want to demonstrate how the cooperative game theory model can be applied in this kind of scenario and demonstrate how beneficial such an analysis is for both parties. The problem that arises is how to decide how to distribute the money at the end of July if the workers agree to collaborate all together. Furthermore, this distribution of the owner must be accepted by all workers in such a way that there are no disputes. This results in a win-win situation for all involved.

The distribution of the 100 unit equally among the workers is one potential solution. Therefore, 25 euros must be given to each employee. From the viewpoints of players 2, 3, and 4, however, this is unfair because player 1 has less power than the other players of every potential coalition.
The following assumptions will be used as the basis for our cooperative game model of this situation:

1. The quantity of the harvest is solely based on the number of employees and their group productivity, and not by any other outside force.
2. The owner requires that at least two employees agree to cooperate due to the nature of the product's harvest.
3. The owner's share of the payment, which is taken from the total sum of 100 unit, is directly inversely proportional to the portion of the production harvest.
4. The goal of each employee is to maximize group productivity and, consequently, increase their individual profits.

The first worker, the second, the third, and the fourth who offered to work are all given the numbers 1, 2, 3, and 4, respectively. We use the specific percentages of harvest from the laborers as values for the characteristic function.
There is a game with set of players $N = \{1,2,3,4\}$ and characteristic function $v: 2^4 \rightarrow \mathbb{R}^+$ such that $v(1) = v(2) = v(3) = v(4) = 0$, $v(12) = 15$, $v(13) = 20$, $v(14) = 17$, $v(23) = 30$, $v(24) = 25$, $v(34) = 35$, $v(123) = 40$, $v(124) = 50$, $v(134) = 40$, $v(234) = 65$ and $v(1234) = 100$
So, we have in total $2^4 - 1 = 15$ subset to be considered in the game before doing the distribution. This game is a superadditive game but not a convex game. It's possible that there isn't a stable solution because the game isn't convex. However, this is not a sufficient condition for the non-existence of a sustainable solution. Since the game is superadditive and essential, a coalition with

more players will always be more beneficial. In this way, it is justified that the employees agreed to form the grand coalition.

We can draw the imputation set of this game. The total sum of 100 unit is allocated among the players, and each profit vector used in the imputation set gives each player a profit higher than what he would obtain if he worked alone. The minimum and maximum profit that can be earned by all players are represented by the vertices of the imputation set. Therefore, the lowest that each player can get is nothing, and the highest is 100 unit. But recall that a worker cannot work alone in the circumstances of our case. Therefore, the solution to this game must be found inside the pyramid, as shown in Figure 1. The imputation set elements have a weakness in that they do not account for the various smaller coalitions that the participants may establish. Therefore, there exist profit vectors in the imputation set that provide the players a lower reward than they would receive if they created a coalition that was smaller than the large coalition. So, we must compress this set of imputations into a more smaller set that only considers imputations that are not dominated by other imputations. Because this manner, all participants accept these kinds of imputations and there are no complaints from anyone. The fact that this game is balanced assures that the core isn't empty. We must resolve the following system in order to determine the game's core.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 100 \\ x_1 + x_2 + x_3 \geq 40 \\ x_1 + x_2 + x_4 \geq 50 \\ x_1 + x_3 + x_4 \geq 40 \\ x_2 + x_3 + x_4 \geq 65 \\ x_3 + x_4 \geq 35 \\ x_2 + x_4 \geq 25 \\ x_2 + x_3 \geq 30 \\ x_1 + x_4 \geq 17 \\ x_1 + x_3 \geq 20 \\ x_1 + x_2 \geq 15 \\ x_1, x_2, x_3 x_4 \geq 0 \end{cases}$$

The core of the game is represented by the set of all these imputations, which is colored red within the set of imputations. The Weber set and the core cover are two additional solution subsets that surround the game's core. The set of all imputations that lie in between the upper and lower vectors is known as the core cover These vectors were determined through calculation and are $\vec{a}$ and $\vec{b}$, respectively.The upper vector is $\vec{b} = (35,60,50,60)$ and the lower is $\vec{a} = (0,0,0,0)$. The Weber set does not include this upper and lower vectors of the core cover. Consequently, the Weber set is a subset of the core cover. Finally, we have the following relationship between the game's core, weber set, and core cover: $C(N,v) \subset CC(N,v) \subset \mathcal{W}(N,v)$. As a result, we conclude that we should only take into account profit vectors that are located in the game's core.

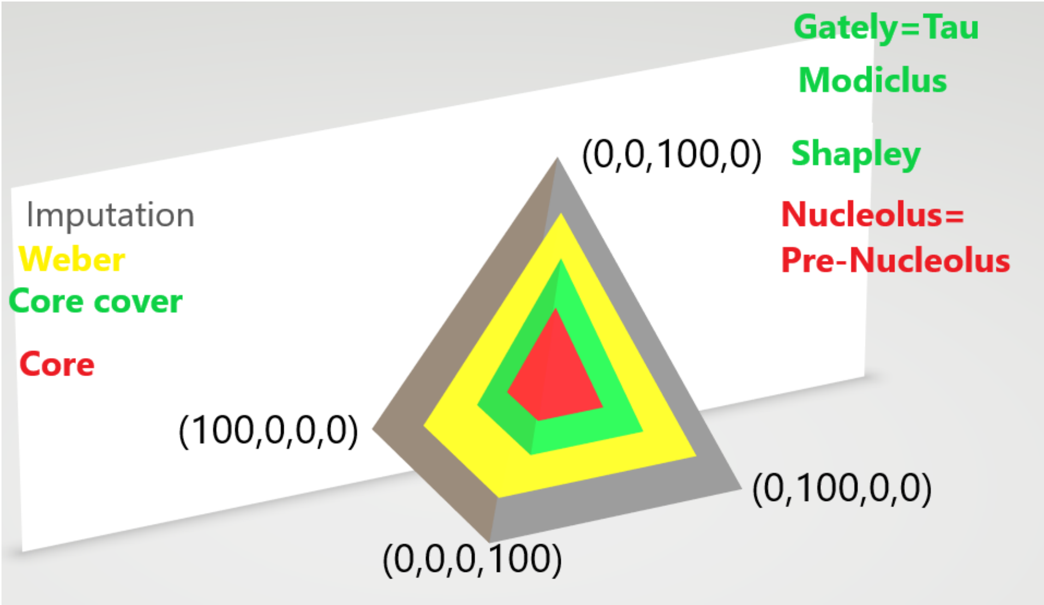

Fig. 1: Schematic presentation of the solutions

We now raise the problem a step further by looking at the single point solutions. Because the game is balanced, we can conclude that it is also quasi-balanced. This guarantees the existence of the tau value and Gately point. The nucleolus exists and is also included in the game's core because it is not empty. The pre-nucleolus, which corresponds to the game's nucleolus, is described in the same way. It makes no difference to the Shapley value whether or not the game is convex or even if the core of the game exists. Formulas make it simple to calculate, and the table below contains the distribution that results from it. The game's core may or may not contain Shapley value.Table 1 shows the percentage distribution for the game's six alternative one-point solutions.

	Player 1	**Player 2**	**Player 3**	**Player 4**
Gately Value	17	**29**	25	29
Modiclus	**18**	28	26	28
Nucleolus	**18**	25	**27**	**30**
Pre-Nucleolus	**18**	25	**27**	**30**
Shapley Value	16	28	**27**	29
Tau-Value	17	**29**	25	29

Table 1: Each employee's percentage of benefits gained based on several concepts

It is determined by compare these six one-point methods that they all produce results that are relatively similar. The Gately point and τ −value exactly coincide. Hence, we must compare four alternative outcomes. At the same time, we have underlined the best values that each of the point solutions provides for each player. Only the nucleolus is located in the game's core; the other three solutions are contained in the core cover set. For a better understanding, we can also create a schematic representation of the point interpretations, as shown below in the figure 1. While we notice the other four values, Gately, Tau, Modiclus, and Shapley, which are located in the core cover.

According to four one-point solution concepts, Figure 2 compares the shares for each employee.

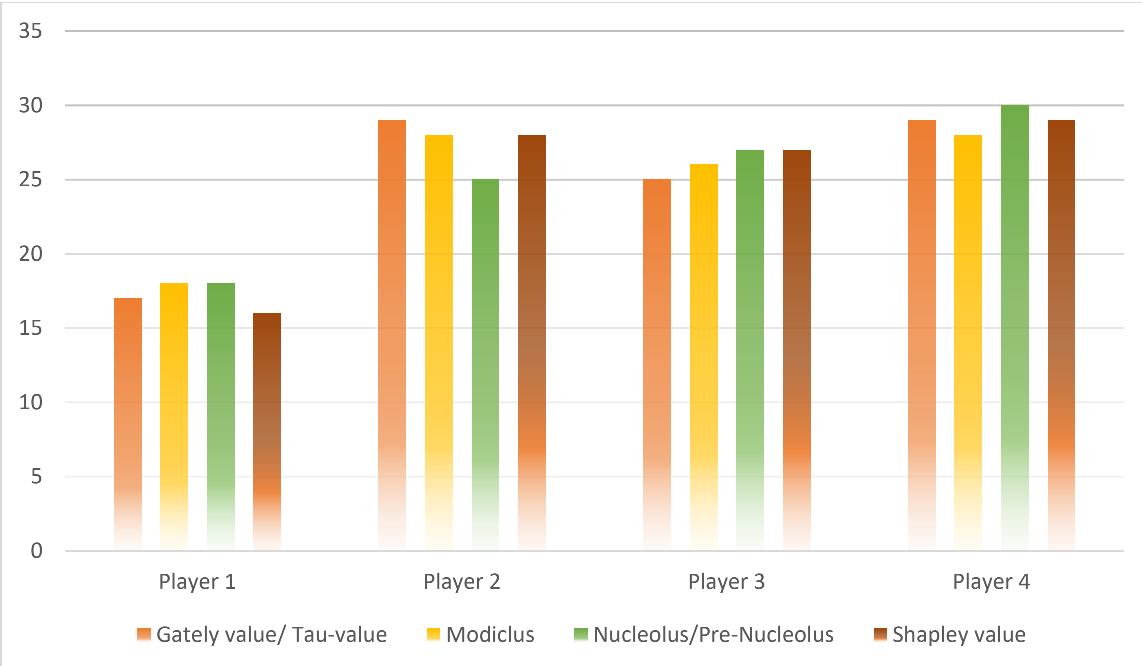

Fig.2: Comparison of allocated rewords using different methods

We find that the four outcomes are highly similar to one another, with only very slight differences between them. For player 1 the distribution according modiclus and the nucleolus/pre-nucleolus are more profitable. The allocations with Gately/Tau value are better for player 2. Player 3 benefits from both the nucleolus and the Shapley value. Finally, player 4 gets more from the Shapley value. Assuming that the four workers will negotiate to reach a final choice, we can conclude that the solution that comes from the nucleolus is the one that is best for everyone . As a result, the vector expressed in percentage with the coordinates $x = (18,25,27,30)$ represents the distribution of the employee's account's 100 unit value. The theoretical study predicted this outcome because the nucleolus was the only point solution found in the game's core part. Theoretically, we demonstrated that the ideal and preferable solutions are those that are present in the game's core.

CONCLUSIONS

In the theoretical reasoning, it is of interest to study the games which are super-additive, because in these games the members are motivated to cooperate. In each situation, cooperation is a key notion since it offers advantages that are beneficial to both the participants and the specific goals. In order to reach a solution as acceptable as possible between the members, the situation modeled as a cooperative game must be essential and balanced. These properties ensure the existence of the elements that are in the most important concept that is the core of the game.

On the other hand, before using this model in a real setting, we must ensure that the members' goals are to increase the coalition's benefits and that they make rational decisions without being influenced by outside factors.

The case study we used as a reference demonstrates to us how easily this model may be used in circumstances involving agreements that can be reached between employees and farmers. By doing this, not only is one of the major obstacles to a fruitful crop's harvest avoided, but each side also stands to gain significantly more from the situation.

Future challenges for us as researchers include developing or applying this model in a proper theoretical framework so that the conclusions drawn are as inclusive as possible and unrestricted by the various values of the characteristic function.

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