DYNAMIC BEHAVIOR OF THE SOLUTIONS FOR A CLASS OF FOUR COUPLED ADVERTISING OSCILLATORS MODEL WITH DELAY

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ABSTRACT

In this paper, a class of four coupled advertising oscillators model with time delay is investigated. By means of mathematical analysis approach, some sufficient conditions to guarantee the stability of the solutions and the existence of oscillatory solutions for the model are obtained. Computer simulations are provided to demonstrate the present results.

Keywords: Coupled advertising oscillator, delay, instability, oscillation.

INTRODUCTION

It is well known that coupled systems can lead some interesting phenomena such as synchronization, chaos, bifurcation and so on. One can find that such coupling systems are widely used in physical, chemical, biological or economical sciences. Therefore, many researchers have investigated various coupled systems [1-17]. For example, Qian et al. considered the following two coupled nonlinear systems with delay coupling [1]:

$$\begin{cases} x_1''(t) + (\gamma x_1^2(t) - \beta)x_1'(t) + \omega_0^2 x_1(t) = \alpha x_2'(t-\tau), \\ x_2''(t) + (\gamma x_2^2(t) - \beta)x_2'(t) + \omega_0^2 x_2(t) = \alpha x_1'(t-\tau). \end{cases} \tag{1}$$

where $\alpha < 0$ is a constant. By means of the multiple scales method, the authors analyzed the distribution of the eigenvalues of its linearized characteristic equations, and obtained the critical value for the occurrence of double Hopf bifurcation. By using the homotopy analysis method, the analytical approximate solutions of the system with parameter values also provided. In order to study the interaction between the number of potential buyers and of the users of two similar products as an effect of the advertising, the nonlinear analysis of two coupled advertising oscillators has been studied by Sterpua and Soreanub [2]:

$$\begin{cases} x_1' = -a(x_1 + bx_2 + 2x_1x_2 + x_2^2 + x_1x_2^2) + c(x_1 - x_3), \\ x_2' = x_1 + x_2 + 2x_1x_2 + x_2^2 + x_1x_2^2, \\ x_3' = -a(x_3 + bx_4 + 2x_3x_4 + x_4^2 + x_3x_4^2) + c(x_3 - x_1), \\ x_3' = x_3 + x_4 + 2x_3x_4 + x_4^2 + x_3x_4^2. \end{cases} \tag{2}$$

where $a, b,$ and c are constants. Rocsoreanu et al. extended model (2) and considered two advertising models linearly coupled via the flow of the potential buyers using two parameters c_1, and c_2:

$$\begin{cases} x_1' = -a(x_1 + bx_2 + 2x_1x_2 + x_2^2 + x_1x_2^2) + c_1(x_1 - x_3), \\ x_2' = x_1 + x_2 + 2x_1x_2 + x_2^2 + x_1x_2^2, \\ x_3' = -a(x_3 + bx_4 + 2x_3x_4 + x_4^2 + x_3x_4^2) + c_2(x_3 - x_1), \\ x_3' = x_3 + x_4 + 2x_3x_4 + x_4^2 + x_3x_4^2. \end{cases} \tag{3}$$

The Hopf bifurcation around the symmetric equilibrium point is performed by using the general formula obtained from the computation of the Lyapunov coefficients [3]. Then Zhang and Zheng have designed three advertising oscillators linked with the number of potential buyers and brand users at moment *t*. A three coupled advertising oscillators model was provided as follows [4]:

$$\begin{cases} x_1' = -a\left(x_1 + bx_2 + 2x_1x_2 + x_2^2 + x_1x_2^2\right) + c(x_1(t-\tau) - x_3(t-\tau)) \\ \quad +c(x_1(t-\tau) - x_5(t-\tau)), \\ x_2' = x_1 + x_2 + 2x_1x_2 + x_2^2 + x_1x_2^2, \\ x_3' = -a\left(x_3 + bx_4 + 2x_3x_4 + x_4^2 + x_3x_4^2\right) + c(x_3(t-\tau) - x_1(t-\tau)) \\ \quad +c(x_3(t-\tau) - x_5(t-\tau)), \\ x_3' = x_3 + x_4 + 2x_3x_4 + x_4^2 + x_3x_4^2, \\ x_5' = -a\left(x_5 + bx_6 + 2x_5x_6 + x_6^2 + x_5x_6^2\right) + c(x_5(t-\tau) - x_1(t-\tau)) \\ \quad +c(x_5(t-\tau) - x_3(t-\tau)), \\ x_6' = x_5 + x_6 + 2x_5x_6 + x_6^2 + x_5x_6^2. \end{cases} \tag{4}$$

where $a > 1,\ b > 1$ and $c > 0.$ By means of the symmetric functional differential equation theories, Pitchfork bifurcation, multiple Hopf bifurcations and multiple branches of bifurcating periodic solutions were obtained. Motivated by the above models, in this paper, we extend model (4) to the following four coupled advertising oscillators:

$$\begin{cases} x_1' = -a_1\left(x_1 + b_1x_2 + 2x_1x_2 + x_2^2 + x_1x_2^2\right) + c_1(x_1(t-\tau) - x_3(t-\tau)) \\ \quad +d_1(x_1(t-\tau) - x_5(t-\tau)) + e_1(x_1(t-\tau) - x_7(t-\tau)), \\ x_2' = x_1 + x_2 + 2x_1x_2 + x_2^2 + x_1x_2^2, \\ x_3' = -a_2\left(x_3 + b_2x_4 + 2x_3x_4 + x_4^2 + x_3x_4^2\right) + c_2(x_3(t-\tau) - x_1(t-\tau)) \\ \quad +d_2(x_3(t-\tau) - x_5(t-\tau)) + e_2(x_3(t-\tau) - x_7(t-\tau)), \\ x_3' = x_3 + x_4 + 2x_3x_4 + x_4^2 + x_3x_4^2, \\ x_5' = -a_3\left(x_5 + b_3x_6 + 2x_5x_6 + x_6^2 + x_5x_6^2\right) + c_3(x_5(t-\tau) - x_1(t-\tau)) \\ \quad +d_3(x_5(t-\tau) - x_3(t-\tau)) + e_3(x_5(t-\tau) - x_7(t-\tau)), \\ x_6' = x_5 + x_6 + 2x_5x_6 + x_6^2 + x_5x_6^2, \\ x_7' = -a_4\left(x_7 + b_4x_8 + 2x_7x_8 + x_8^2 + x_7x_8^2\right) + c_4(x_7(t-\tau) - x_1(t-\tau)) \\ \quad +d_4(x_7(t-\tau) - x_3(t-\tau)) + e_4(x_7(t-\tau) - x_5(t-\tau)), \\ x_8' = x_7 + x_8 + 2x_7x_8 + x_8^2 + x_7x_9^2. \end{cases} \tag{5}$$

where $a_i > 1,\ \ b_i > 1,$ and $c_i \neq 0,\ d_i \neq 0,\ e_i \neq 0\ (i = 1, \cdots, 4)$ are real numbers. By means of mathematical analysis method, the periodic solution of system (5) has been provided. The computer simulation is provided to support the present theoretical result. We point out that the bifurcating method is very hard to deal with system (5) because there are twenty parameters in system (5).

PRELIMINARIES

Obviously, system (5) has an equilibrium point (0, 0, ⋯, 0). The linearized system of model (5) around the zero equilibrium point is the following:

$$\begin{cases} x_1' = -a_1x_1 - a_1b_1x_2 + c_1(x_1(t-\tau) - x_3(t-\tau)) + d_1(x_1(t-\tau) - x_5(t-\tau)) \\ \qquad +e_1(x_1(t-\tau) - x_7(t-\tau)), \\ \qquad x_2' = x_1 + x_2, \\ x_3' = -a_2x_3 - a_2b_2x_4 + c_2(x_3(t-\tau) - x_1(t-\tau)) + d_2(x_3(t-\tau) - x_5(t-\tau)) \\ \qquad +e_2(x_3(t-\tau) - x_7(t-\tau)), \\ \qquad x_3' = x_3 + x_4, \\ x_5' = -a_3x_5 - a_3b_3x_6 + c_3(x_5(t-\tau) - x_1(t-\tau)) + d_3(x_5(t-\tau) - x_3(t-\tau)) \\ \qquad +e_3(x_5(t-\tau) - x_7(t-\tau)), \\ \qquad x_6' = x_5 + x_6, \\ x_7' = -a_4x_7 - a_4b_4x_8 + c_4(x_7(t-\tau) - x_1(t-\tau)) + d_4(x_7(t-\tau) - x_3(t-\tau)) \\ \qquad +e_4(x_7(t-\tau) - x_5(t-\tau)), \\ \qquad x_8' = x_7 + x_8. \end{cases} \tag{6}$$

Thus, the system (5) can be expressed in the following matrix form:

$$X'(t) = AX(t) + BX(t-\tau) + \Phi(X(t)) \tag{7}$$

where $X(t) = [x_1(t),\ x_2(t),\ \cdots,\ x_8(t)]^T,\ X(t-\tau) = [x_1(t-\tau),\ 0,\ \cdots,\ x_7(t-\tau), 0]^T$,
$\Phi(X(t)) = [-2a_1x_1x_1 - a_1x_2^2 - a_1x_1x_2^2, 2x_1x_2 + x_2^2 + x_1x_2^2, \cdots, 2x_7x_8 + x_8^2 + x_7x_8^2]^T$.
Both $A = (a_{ij})$ and $B = (b_{ij})$ are eight by eight matrices as follows.

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & a_{34} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{78} \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

where $a_{11} = -a_1, a_{12} = -a_1b_1, a_{33} = -a_3, a_{34} = -a_3b_3,\ \cdots, a_{78} = -a_4b_4$.

$$B = (b_{ij}) = \begin{pmatrix} b_{11} & 0 & -c_1 & 0 & -d_1 & 0 & -e_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_2 & 0 & b_{33} & 0 & -d_2 & 0 & -e_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -c_4 & 0 & -d_4 & 0 & -e_4 & 0 & b_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $b_{11} = c_1 + d_1 + e_1, b_{33} = c_2 + d_2 + e_2,\ b_{55} = c_3 + d_3 + e_3, b_{77} = c_4 + d_4 + e_4$.

Apparently, the matrix form of system (6) is the follows:

$$X'(t) = AX(t) + BX(t-\tau) \tag{8}$$

Lemma 1 Assume that $a_i > 1,\ b_i > 1,$ and $c_i \neq 0,\ d_i \neq 0,\ e_i \neq 0\ (i = 1,\cdots,4)$ are real numbers, and the matrix $C\ (= A + B\ )$ is a nonsingular matrix, then system (8) has a unique equilibrium, implying that system (5) (or (7)) has a unique equilibrium. Namely, the zero equilibrium point.

Proof An equilibrium point $x^* = [x_1^*,\ x_2^*,\ \cdots,\ x_8^*]^T$ of system (8) is a constant solution of the following algebraic equation

$$Ax^* + Bx^* = (A+B)x^* = Cx^* = 0 \tag{9}$$

Since $C\ (= A + B\ )$ is an eight by eight matrix, according to the linear algebraic knowledge, if C is a singular matrix, Eq. (9) may have many infinity solutions. However, if C is a nonsingular matrix, Eq. (9) has only one solution, namely, the trivial solution. Noting that in system (7), the nonlinear term is $\Phi(X)$, and if and only if $\Phi(\mathbf{0}) = \mathbf{0}$. Therefore, system (8) has a unique equilibrium point suggests that system (5) (or (7)) has a unique trivial solution.

Lemma 2 Assume that $a_i > 1,\ b_i > 1,$ and $c_i \neq 0,\ d_i \neq 0,\ e_i \neq 0\ (i = 1,\cdots,4)$ are real numbers. All solutions of system (5) (or (7)) are uniformly bounded.

Proof To prove the boundedness of the solutions in system (5), we construct a Lyapunov function $V(t) = \sum_{i=1}^{8} \frac{1}{2} x_i^2(t)$. Calculating the derivative of $V(t)$ through system (5) one get:

$$\begin{aligned}
&V'(t)|_{(5)} = \textstyle\sum_{i=1}^{8} x_i(t)x_i'(t)\\
&= (1-a_1)x_1^2 + (1-a_1b_1)x_1x_2 + (1-2a_1)x_1^2x_2 + (1-a_1)x_1x_2^2 + (1-a_1)x_1^2x_2^2\\
&\quad + (c_1 + d_1 + e_1)x_1^2 - c_1x_1x_3 - d_1x_1x_5 - e_1x_1x_7 + \cdots\\
&+(1-a_4)x_7^2 + (1-a_4b_4)x_7x_8 + (1-2a_4)x_7^2x_8 + (1-a_4)x_7x_8^2 + (1-a_4)x_7^2x_8^2\\
&\quad +(c_4 + d_4 + e_4)x_7^2 - c_4x_7x_1 - d_4x_7x_3 - e_4x_7x_5
\end{aligned} \tag{10}$$

Noting that $1 - a_i < 0$ since $a_i < 1\ (i = 1,\cdots,4)$. When $x_i \to +\infty\ (i = 1,\cdots,8)$, $x_1^2x_2^2,\ \cdots,$ $x_7^2x_8^2$ are higher order infinity than x_i^2 and $x_ix_{i+1}(i = 1,\cdots,8)$. Therefore, there exists suitably large $L > 0$ such that $V'(t)|_{(5)} < 0$ as $x_i > L\ (i = 1,\cdots,8)$. This means that all solutions of system (5) (or (7)) are bounded.

MAIN RESULTS

Theorem 1 Assume that $a_i > 1,\ b_i > 1,$ and $c_i \neq 0,\ d_i \neq 0,\ e_i \neq 0\ (i = 1,\cdots,4)$ are real numbers,, and system (5) has a unique equilibrium point, for selecting parameter values. Let $\alpha_1, \alpha_2, \cdots, \alpha_8$ and $\beta_1, \beta_2, \cdots, \beta_8$ be eigenvalues of matrices A and B, respectively. If Re $(\alpha_i + \beta_i) < -c < 0\ \ (i = 1,\cdots,8)$. Then the unique equilibrium point of system (5) is asymptotically stable.

Proof Since Re $(\alpha_i + \beta_i) < -c < 0\ (i = 1,\cdots,8)$, there exists $M \geq 1$such that $\left\| e^{(A+B)t} \right\| \leq Me^{-ct}$. We first show that the trivial solution of system (8) is asymptotically stable. Rewrite system (8) in the form

$$\begin{aligned}
X'(t) &= (A+B)X(t) - B\int_{t-\tau}^{t} X'(s)\,ds\\
&= (A+B)X(t) - B\textstyle\int_{t-\tau}^{t}[AX(s) + BX(t-\tau)]\,ds, t \geq \tau
\end{aligned} \tag{11}$$

By variation of parameter we have

$$X(t) = e^{(A+B)(t-\tau)}X(\tau) - \int_{\tau}^{t} ds \int_{s-\tau}^{s} e^{(A+B)(t-s)} B[AX(u) + BX(u-\tau)]du, t \geq \tau \quad (12)$$

Hence for , $t \geq \tau$ we have

$$\|X(t)\| \leq KMe^{-c(t-\tau)} + M\|B\| \int_{\tau}^{t} ds \int_{s-\tau}^{s} e^{-c(t-s)} (\|A\| \, \|X(u)\| + \|B\| \, \|X(u-\tau)\|)du \quad (13)$$

where $K = \sup_{t\in[-\tau,\tau]} \|X(t)\|$ and $\|A\| = \max_{1\leq j\leq 8} \sum_{i=1}^{8} |a_{ij}|$. Now we consider

$$\|Y(t)\| = KMe^{-c(t-\tau)} + M\|B\| \int_{\tau}^{t} ds \int_{s-\tau}^{s} e^{-c(t-s)} (\|A\| \, \|Y(u)\| + \|B\| \, \|Y(u-\tau)\|)du \quad (14)$$

Obviously we have $\|X(t)\| \leq \|Y(t)\|$. We shall prove that there exists a positive number $\sigma\ (< c)$ such that $\|Y(t)\| = KMe^{-\sigma(t-\tau)},\ t \geq \tau$. Indeed,

$$\begin{aligned}
&KMe^{-c(t-\tau)} + M\|B\| \int_{\tau}^{t} ds \int_{s-\tau}^{s} e^{-c(t-s)} (\|A\| \, \|Y(u)\| + \|B\| \, \|Y(u-\tau)\|)du \\
&= KMe^{-c(t-\tau)} + M\|B\| \int_{\tau}^{t} ds \int_{s-\tau}^{s} e^{-c(t-s)} \left(\|A\| \, KMe^{-\sigma(t-\tau)} + \|B\| \, KMe^{-\sigma(t-2\tau)}\right)du \\
&= KMe^{-c(t-\tau)} + \|B\| \frac{KM^2(\|A\| \, e^{\sigma\tau} + \|B\|e^{2\sigma\tau})}{-\sigma} \int_{\tau}^{t} e^{-c(t-s)} \left(e^{-\sigma s} - e^{-\sigma(s-\tau)}\right)ds \\
&= KMe^{-c(t-\tau)} + \|B\| \frac{KM^2(\|A\| \ + \|B\|e^{\sigma\tau})(e^{\sigma\tau} - 1)}{\sigma(c-\sigma)} e^{-ct}e^{\sigma\tau}\left(e^{(c-\sigma)t} - e^{(c-\sigma)\tau}\right) \\
&= KMe^{-c(t-\tau)} + \|B\| \frac{KM^2(\|A\| \ + \|B\|e^{\sigma\tau})(e^{\sigma\tau} - 1)}{\sigma(c-\sigma)} \left(e^{-\sigma(t-\tau)} - e^{-c(t-\tau)}\right)
\end{aligned} \quad (15)$$

Noting that $\lim_{\sigma\to 0+} e^{\sigma\tau} = 1$. Therefore, one can select a positive constant $\sigma\ (< c)$ such that $\frac{M\|B\|(\|A\| + \|B\|e^{\sigma\tau})(e^{\sigma\tau}-1)}{\sigma(c-\sigma)} = 1$, then

$$\begin{aligned}
&KMe^{-c(t-\tau)} + \|B\| \frac{KM^2(\|A\| \ + \|B\|e^{\sigma\tau})(e^{\sigma\tau} - 1)}{\sigma(c-\sigma)} \left(e^{-\sigma(t-\tau)} - e^{-c(t-\tau)}\right) \\
&= KMe^{-c(t-\tau)} + KM\left(e^{-\sigma(t-\tau)} - e^{-c(t-\tau)}\right) \\
&= KMe^{-\sigma(t-\tau)} = Y(t)
\end{aligned} \quad (16)$$

From (16) we have $Y(t) \to 0$ as $t \to +\infty$, implying that have $X(t) \to 0$ as $t \to +\infty$. This means that the trivial solution in system (8) is asymptotically stable. Now for system (5), the nonlinear terms are higher order infinitesimal when $x_i \to 0\ (i = 1, \cdots, 8)$. Thus, the asymptotic stability of the trivial solution in system (8) implies that the trivial solution in system (5) is asymptotically stable. The proof is completed.

Theorem 2 Assume that $a_i > 1$, $b_i > 1$, and $c_i \neq 0$, $d_i \neq 0$, $e_i \neq 0$ $(i = 1, \cdots, 4)$ are real numbers,, and system (5) has a unique equilibrium point, for selecting parameter values. Let $\alpha_1, \alpha_2, \cdots, \alpha_8$ and $\beta_1, \beta_2, \cdots, \beta_8$ be eigenvalues of matrices A and B, respectively. If there exists some α_i such that $\alpha_i > 0$ $(\mathrm{Re}\,(\alpha_i) > 0)$ or $\mathrm{Re}\,(\beta_i) > |\,\mathrm{Re}\,(\alpha_i)|$ $(i \in \{1, \cdots, 8\})$. Then the unique equilibrium point of system (5) is unstable. System (5) generates a limit cycle. Namely, there is a periodic solution.

Proof Obviously, system (5) (or (7)) has a unique unstable zero equilibrium point if and only if system (8) has a unique unstable zero equilibrium point. Therefore, in the following we consider the instability of the unique zero equilibrium point in system (8). Noting that $\alpha_1, \alpha_2, \cdots, \alpha_8$ and $\beta_1, \beta_2, \cdots, \beta_8$ are eigenvalues of matrices A and B, respectively. Then the characteristic equation of system (8) is

$$\det\left[\lambda I_{ij} - a_{ij} - b_{ij} e^{-\lambda \tau}\right] = 0 \tag{17}$$

where I_{ij} is an eight by eight identity matrix. Namely,

$$\prod_{i=1}^{8} \left(\lambda - \alpha_i - \beta_i e^{-\lambda \tau}\right) = 0 \tag{18}$$

Noting that B has a zero eigenvalue. Form (18) we consider the following equation for some $j \in \{1, \cdots, 8\}$.

$$\lambda - \alpha_j - \beta_j e^{-\lambda \tau} = 0 \tag{19}$$

We select some α_j such that $\alpha_j > 0$ $(\mathrm{Re}\left(\alpha_j\right) > 0)$ and $\beta_j = 0$. Thus, $\lambda = \alpha_j$. If $\mathrm{Re}\,(\beta_i) > |\,\mathrm{Re}\,(\alpha_i)|$, this means that the characteristic equation (19) has a positive real root or a complex root with positive real part. Therefore, the trivial solution of system (8) is unstable, implying that the trivial solution of system (5) (or (7)) is unstable. Since system (5) has a unique unstable equilibrium point, all solutions of system (5) are bounded which will force system (5) to generate a limit cycle, namely, a periodic solution [18, 19].

For convenience, let $\mu(A) = \max\limits_{1 \le j \le 8} (a_{jj} + \sum_{i=1}^{8} |a_{ij}|)$ [20]. Then we have

Theorem 3 Assume that $a_i > 1$, $b_i > 1$, and $c_i \neq 0$, $d_i \neq 0$, $e_i \neq 0$ $(i = 1, \cdots, 4)$ are real numbers,, and system (5) has a unique equilibrium point, for selecting parameter values. If

$$\mu(A) > \|B\| \tag{20}$$

holds, then the unique equilibrium point of system (5) is unstable. System (5) generates a limit cycle. Namely, there is a periodic solution.

Proof We shall show that the unique equilibrium of system (8) is unstable. Let $z(t) = \sum_{i=1}^{8} |x_i(t)|$. Then we have from system (8)

$$\frac{dz(t)}{dt} \le \mu(A) z(t) + \|B\| z(t - \tau) \tag{21}$$

Specially, for equation

$$\frac{dw(t)}{dt} = \mu(A) w(t) + \|B\| w(t - \tau) \tag{22}$$

If the unique equilibrium of system (22) is unstable, then the characteristic equation associated with (22) given by

$$\lambda = \mu(A) + \|B\| e^{-\lambda \tau} \tag{23}$$

will have a real positive root. Noting that equation (23) is a transcendental equation, the characteristic values may be complex numbers. However, we claim that equation (23) has a real positive root. Let

$$f(\lambda) = \lambda - \mu(A) - \|B\|e^{-\lambda\tau} \tag{24}$$

Obviously, $f(\lambda)$ is a continuous function of λ. Noting that $\mu(A) > 0$, $\|B\|$ is bounded, and $f(0) = -\mu(A) - \|B\| < 0$. On the other hand, there exists a suitably large positive number N such that $f(N) = N - \mu(A) - \|B\|e^{-N\tau} > 0$. According to the Intermediate Value Theorem of continuous function, there exists a positive number n such that $f(n) = 0$ $\left(n \in (0, N)\right)$. In other words, there exists a positive characteristic root of equation (23). Therefore, the trivial solution of system (22) is unstable. Similar to Theorem 2, system (5) generates a limit cycle, namely, a periodic solution.

SIMULATION RESULT

This simulation is based on system (5). First we select parameters $a_1 = 3.5,\ a_2 = 3.6,\ a_3 = 3.45, a_4 = 4.8;\quad b_1 = 3.15,\ b_2 = 3.16,\ b_3 = 3.17, b_4 = 5.18;\ c_1 = 0.78,\ c_2 = 0.45,\ c_3 = -0.24, c_4 = 0.65;\ d_1 = -0.35,\ d_2 = 1.25,\ d_3 = -0.15, d_4 = 0.25;\ e_1 = -1.98,\ e_2 = -0.35,\ e_3 = 1.65, e_4 = -1.75$. Thus, the eigenvalues of matrices A and B are $-1.2500 \pm 2.4418\, i,\ -1.3000 \pm 2.4670\, i, -1.2250 \pm 2.4453\, i, -1.9000 \pm 4.0559\, i,$ and $-0.4969 \pm 1.4564\, i, -1.5495,\ 0.0933,\ 0,\ 0,\ 0,\ 0,$ respectively. Even if there is a positive root 0.0933
of matrix B, the conditions of Theorem 1 are still satisfied since $-1.2250 + 0.0933 = -1.1217 < 0$. The trivial solution is stable in system (5) (see figure 1 and figure 2). Then we select $a_1 = 1.5,\ a_2 = 1.6,\ a_3 = 1.45, a_4 = 1.2;\quad b_1 = 2.15,\ b_2 = 2.15,\ b_3 = 2.35, b_4 = 2.55;\ c_1 = 0.55,\ c_2 = -0.65,\ c_3 = -0.24, c_4 = 0.45;\ d_1 = -0.35,\ d_2 = 0.25,\ d_3 = -0.65, d_4 = 0.35;\ e_1 = -0.98,\ e_2 = -0.35,\ e_3 = 0.65, e_4 = -0.25$. The eigenvalues of matrices A and B are $-0.3000 \pm 1.3229\, i,\ -0.2500 \pm 1.4534\, i, -0.2250 \pm 1.3854\, i, -0.1000 \pm 1.3601\, i,$ and $-0.6001 \pm 0.6405\, i,\ 0.1885,\ 0.0217,\ 0,\ 0,\ 0,\ 0,$ respectively. Obviously, $0.1885 > |0.1000|$. Based on Theorem 2, system (5) generates a periodic solution (see figure 3 and figure 4). Finally we select parameters $a_1 = 3.65,\ a_2 = 3.12,\ a_3 = 3.25, a_4 = 3.35;\quad b_1 = 2.65,\ b_2 = 2.12,\ b_3 = 2.75, b_4 = 2.85;\ c_1 = -1.45,\ c_2 = 0.65,\ c_3 = -0.95, c_4 = 1.95;\ d_1 = -0.85,\ d_2 = 1.35,\ d_3 = -1.15, d_4 = 0.25;\ e_1 = -1.68,\ e_2 = 0.45,\ e_3 = 0.25, e_4 = -1.65.$ Then $\mu(A) = 8.6725$, and $\|B\| = 5.1000$. The conditions of Theorem 3 hold. We see that there is a periodic solution in system (5) (see figure 5 and figure 6).

Fig. 1 Convergence of the solutions, delay: 1.25.

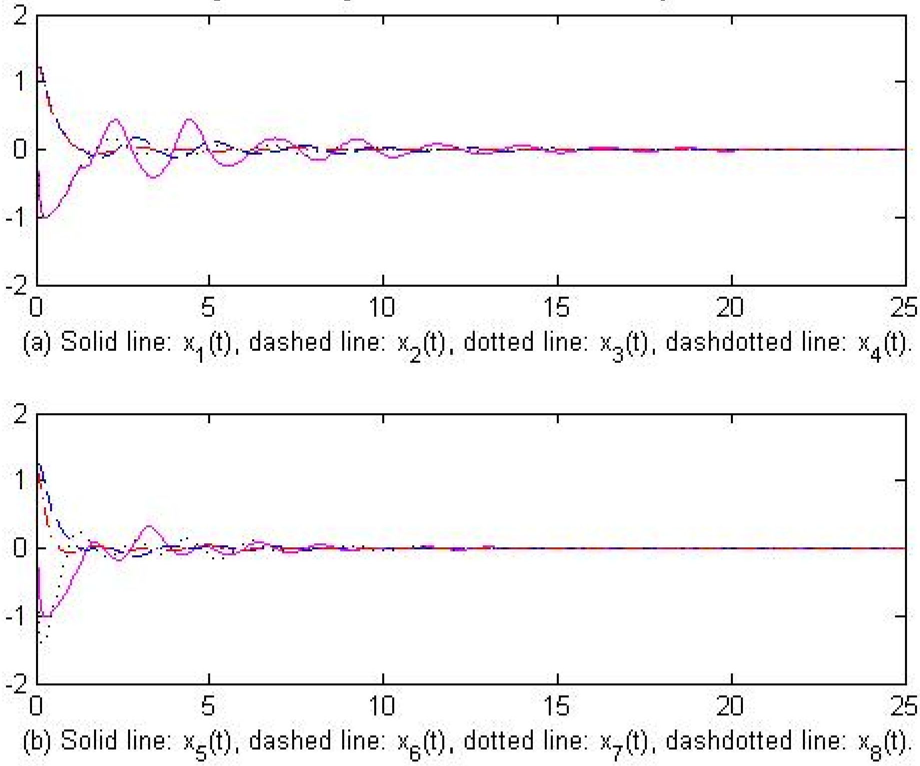


Fig. 2 Convergence of the solutions, delay: 3.25.

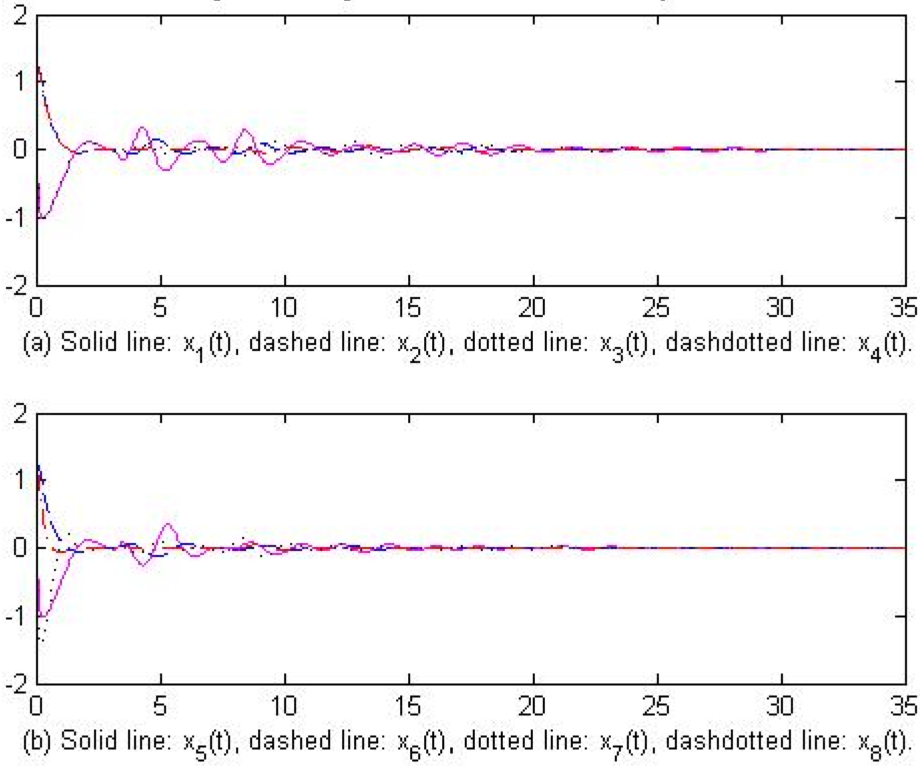
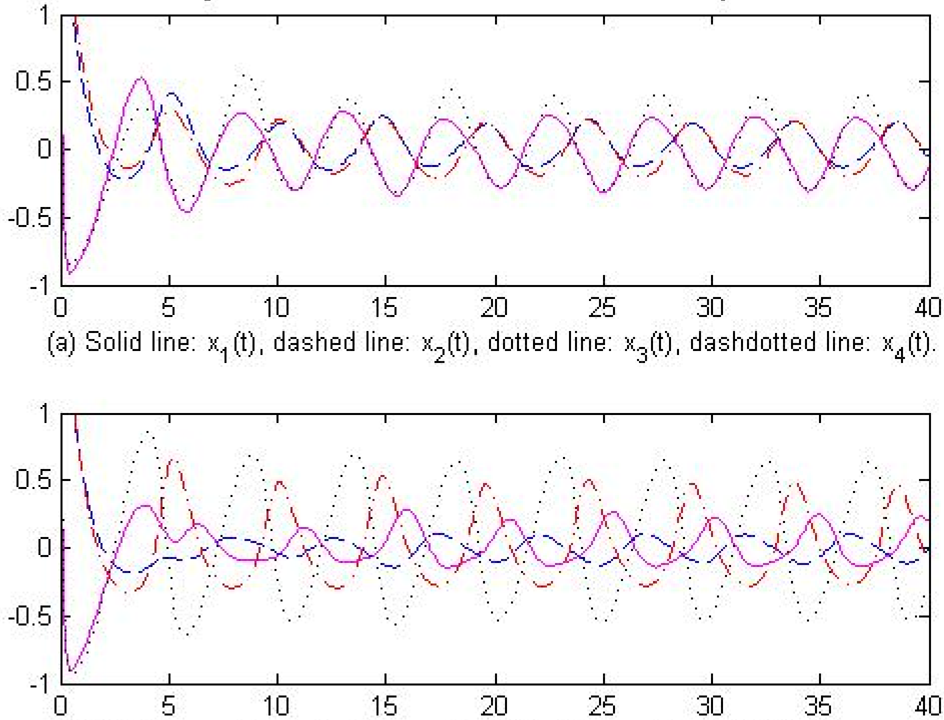


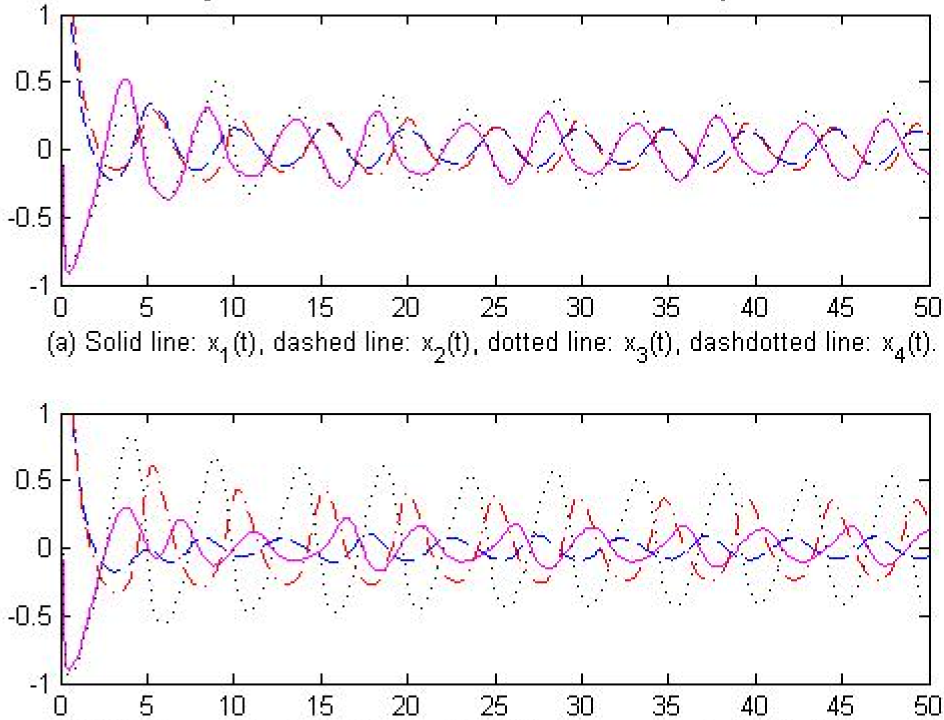
Fig. 3 Periodic oscillation of the solutions, delay: 0.45.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$, dashdotted line: $x_4(t)$.

(b) Solid line: $x_5(t)$, dashed line: $x_6(t)$, dotted line: $x_7(t)$, dashdotted line: $x_8(t)$.

Fig. 4 Periodic oscillation of the solutions, delay: 0.85.



(a) Solid line: $x_1(t)$, dashed line: $x_2(t)$, dotted line: $x_3(t)$, dashdotted line: $x_4(t)$.

(b) Solid line: $x_5(t)$, dashed line: $x_6(t)$, dotted line: $x_7(t)$, dashdotted line: $x_8(t)$.

Fig. 5 Periodic oscillation of the solutions, delay: 0.35.

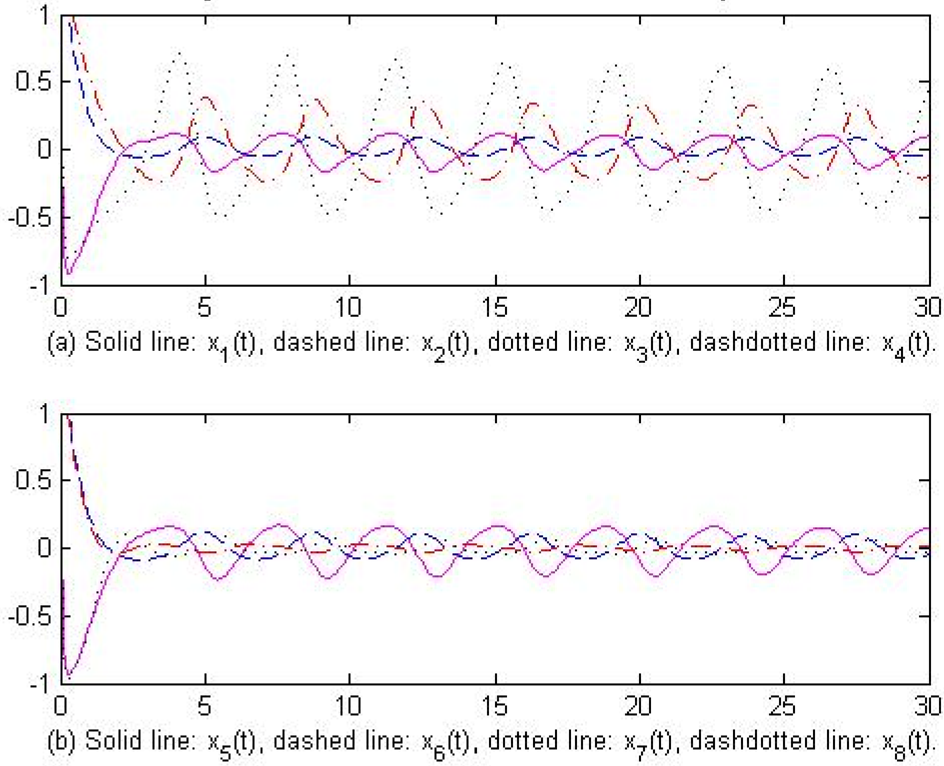
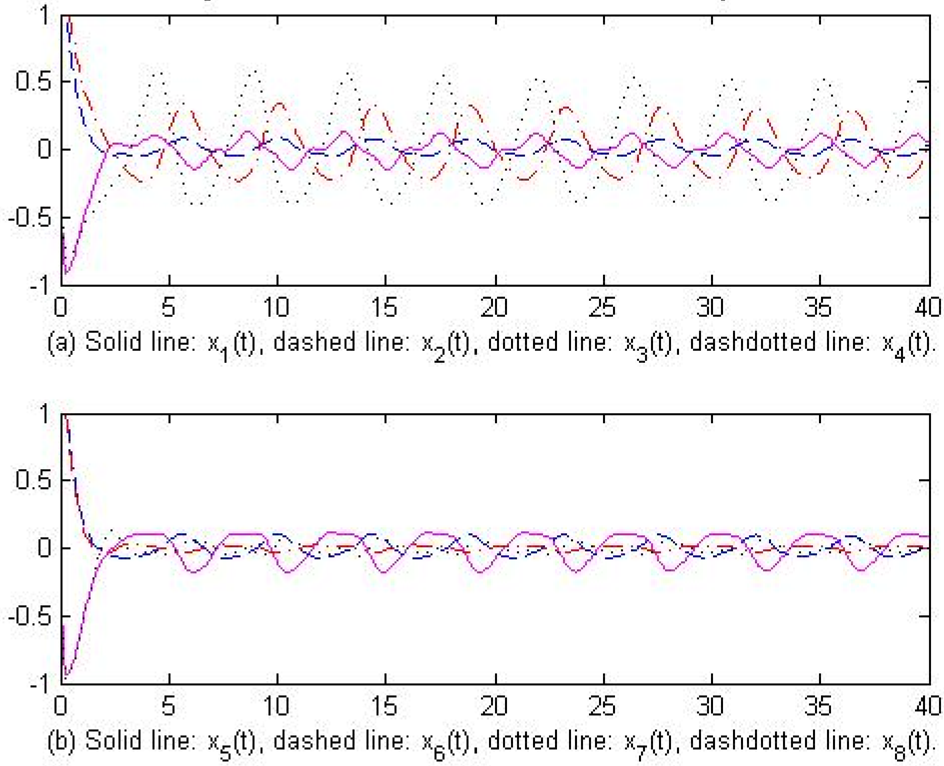


Fig. 6 Periodic oscillation of the solutions, delay: 0.65.



ISSN 2059-9951

CONCLUSION

In this paper, we have discussed the dynamical behavior of four coupled advertising oscillator model with delay. Based on the mathematical analysis theory, a simple criterion to guarantee the existence of periodic oscillations, which is easy to check, as compared to the bifurcating method has been proposed. Some simulations are provided to indicate the correctness of the criterion. Our simulation indicates that the present criteria only are sufficient conditions.

REFERENCES

[1] Qian, Y.H., Fu, H.X., & Guo, J.M. (2019) Weakly resonant double Hopf bifurcation in coupled nonlinear systems with delayed feedback and application of homotopy analysis method, *J. Low Freq. Noise, Vib. Act. Control*, 38(3): 1651-1675.
[2] Sterpua, M., & Soreanub, C. (2005) Hopf bifurcation in a system of two coupled advertising oscillators, *Nonlinear Anal. Real World Appl*., 6:1-12.
[3] Rocsoreanu, C. et al. (2009) Lyapunov coefficients for non-symmetrically coupled identical dynamical systems: Application to coupled advertising models, *Disc. Cont. Dyn. Syst*., 11(3):785-803.
[4] Zhang, C.R., & Zheng, H.F. (2011) D_3-Equivariant coupled advertising oscillators model,
Commun Nolinear Sci Numer Simulat., 16: 1706-1711.
[5] Ge, Z.M. et al. (2008) Pragmatical adaptive chaos control from a new double van der Pol system to a new double Duffing system, *Appl. Math. Comput*., 203(2): 513-522.
[6] Qian, Y.H., & Chen, S.M. (2010) Accurate approximate analytical solutions for multi-degree-of-freedom coupled van der Pol-Duffing oscillators by homotopy analysis method,
Commun. Nonlinear Sci. Numer. Simu., 15(10): 3113-3130.
[7] Rompala, K., Rand, R., & Howland, H. (2007) Dynamics of three coupled van der Pol oscillators with application to circadian rhythms, *Commun. Nonlinear Sci. Numer. Simu*., 12(5): 794-803.
[8] Kadji, H.G., Orou, J.B., & Woafo, P. (2008) Synchronization dynamics in a ring of four mutually coupled biological systems, *Commun. Nonlinear Sci. Numer. Simu*., 13 (7): 1361-1372.
[9] Lu, L.L. et al. (2019) Phase synchronization and mode transition induced by multiple time delays and noises in coupled FitzHugh-Nagumo model, *Physica A: Stat. Mech. Appl*., 535, 122419.
[10] Wang, Z. et al. (2020) Delay-induced synchronization in two coupled chaotic memristive Hopfield neural networks, *Chaos, Solitons, Fractals*, 134, 109702.
[11] Meng, H. et al. (2020) The generalization of equal-peak method for delay-coupled nonlinear system, *Physica D: Nonlinear Phenomena*, 403, 132340.
[12] Wang, W.Y., & Chen, L.J. (2015) Weak and non-resonant double Hopf bifurcations in m coupled van der Pol oscillators with delay coupling, *Appl. Math. Model*., 39(10-11): 3094-3102.
[13] Ermakov, I.V., Sande, G.V., & Danckaert, J. (2012) Semiconductor ring laser subject to delayed optical feedback: Bifurcations and stability, *Commun. Nonlinear Sci. Numer. Simu.,* 17(12): 4767-4779.
[14] Jiang, H.B., Bi, Q.S., & Zheng, S. (2012) Impulsive consensus in directed networks of identical nonlinear oscillators with switching topologies, Commun. Nonlinear Sci. Numer. Simu., 17(1): 378-387.
[15] Li, Y.Q., Jiang, W.H., & Wang, H.B. (2012) Double Hopf bifurcation and quasi-periodic attractors in delay-coupled limit cycle oscillators, *J. Math. Anal. Appl*., 387(2): 1114-1126.

ISSN 2059-9951

[16] Raaj, A., Mondal, S., & Jagdish, V. (2021) Investigating amplitude death in a coupled nonlinear aeroelastic system, *Int. J. Non-Linear Mech*., 129, 103659.
[17] Shen, Z.L., & Zhang, C.R. (2014) Double Hopf bifurcation of coupled dissipative Stuart-Landau oscillators with delay, *Appl. Math. Comput*., 227(15): 553-566.
[18] Chafee, N. (1971) A bifurcation problem for a functional differential equation of finitely retarded type, J. Math. Anal. Appl., 35(2): 312-348.
[19] Feng,, & Plamondon, R. (2012) An oscillatory criterion for a time delayed neural ring network model, *Neural Networks*, 29: 70-79.
[20] Gopalsamy, K (1992) *Stability and oscillations in delay differential equations of populations dynamics,* Kluwer Academic Publishers.