FIXED POINT RESULTS FOR F - CONTRACTION MAPPING IN MODULAR METRIC SPACES

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Abstract

In this paper, we present the concept of F -contraction mapping on modular metric spaces and prove a fixed point theorem of contraction mapping. The relative results will be given. The obtained results generalized and extend fixed point results of modular metric spaces in the existing literature.

Keywords*:* modular metric spaces, F -contraction, fixed point theory

1. Introduction

Throughout this paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ will denote the set of natural numbers, real numbers and positive real numbers.

Fixed point theory is one of the most useful results in a variety of areas such as nonlinear analysis, differential equation, operator theory, etc. In 1922, Banach first proved formular and proved a theorem regarding a contraction mapping, known as Banach contraction principle(see [1]). Due to it's application in mathematic, several authors have obtained many interesting extensions and generalization of the Banach contraction principle(see [2, 3, 4, 5]).

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Sub-sequently, Wardowski[6] introduce a new definition called F -contraction and proved a new theorem concerning F-contraction in 2012. Wardowski defined the F -contraction as follows.

Definition 1.1. [6] Let $\mathcal{F}$ be the family of all functions $F : \mathbb{R}_+ \to \mathbb{R}$ such that: :

(F1) F is strictly increasing, *i.e.* for all $x, y \in \mathbb{R}_+$ such that x < y, $F(x) < F(y)$;

(F2) for each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

(F3) there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Definition 1.2. [6] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F -contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y))$$

A new generalization of Banach contraction principle haven been given by Wardowski as follows.

Theorem 1.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n X\}_{n=1}^{\infty}$ converges to x^*.*

This paper is to consider and establish fixed point theorem of F -contraction on modular metric space. The results of this paper generalize and improve some known results respect to F -contraction mapping from literature.

2. Preliminaries

Chistyakov introduce the notion of modular metric space in 2006(see[7, 8, 9]). Chistyakov defined modular metric space as follows.

Let X be a nonempty set. Throughout this paper, for a function $w : (0, \infty) \times X \times X \to [0, \infty)$, we write

$$w_\lambda(x, y) = w(\lambda, x, y),$$

for all $\lambda > 0$ and $x, y \in X$.

Definition 2.1. [9] Let X be a nonempty set. A function $w : (0, \infty) \times X \times X \to [0, \infty)$ is said to be a metric modular on X if it satisfies, for all $x, y, z \in X$, the following condition:

(i) $w_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

(ii) $w_\lambda(x, y) = w_\lambda(y, x)$ for all $\lambda > 0$;

(iii) $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y)$ for all $\lambda, \mu > 0$.

If instead of (i) we have only the condition (i')

$$w_\lambda(x, x) = 0 \textit{ for all } \lambda > 0, x \in X,$$

then w is said to be a pseudomodular (metric) on X.

An important property of the (metric) pseudomodular on set X is that the mapping $\lambda \mapsto w_\lambda(x, y)$ is non increasing for all $x, y \in X$.

Definition 2.2. [9] Let w is a pseudomodular on X. Fixed $x_0 \in X$. The set

$$\{X_w = X_w(x_0) = \{x \in X : w_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty \}$$

is said to be a modular metric space (around x_0).

Definition 2.3. [10] Let X_w be a modular metric space.

(i) The sequence $\{x_n\}_{n\in N}$ in X_w is said to be w-convergent to $x \in X_w$ if and only if $w_\lambda(x_n, x) \longrightarrow 0$, as $n \longrightarrow \infty$ for some $\lambda > 0$;

(ii) The sequence $\{x_n\}_{n\in N}$ in X_w is said to be w-Cauchy if $w_\lambda(x_m, x_n) \longrightarrow 0$ as $m, n \longrightarrow \infty$ for some $\lambda > 0$;

(iii) A subset C of X_w is said to be w-complete if any w-Cauchy sequence in C is a convergent sequence and its limit is in C.

3. Main Results

Definition 3.1. Let $\mathfrak{F}$ be the family of all functions $F : \mathbb{R}_+ \longrightarrow \mathbb{R}$ which satisfy conditions (F1) and (F2).

Theorem 3.2. *Let w be metric modular on X and X_w be a w-complete modular metric space induced by w. Let $T : X_w \longrightarrow X_w$ and $F \in \mathfrak{F}$ satisfy that if*

$$w_\lambda(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(w_\lambda(Tx, Ty)) \leq F(w_\lambda(x, y)) \qquad (1)$$

for all $x, y \in X_w$ and for all $\lambda > 0$, then T has a unique fixed point in X_w.

Proof. First, we can get that T has at most one fixed point. Indeed, if $x_1^* x_2^* \in X$, $Tx_1^* = x_1^* \neq x_2^* = Tx_2^*$, then we obtain

$$\tau \leq F(w_\lambda(x_1^*, x_2^*)) - F(w_\lambda(Tx_1^*, Tx_2^*)),$$

for all $\lambda > 0$. Which is a contradiction.

In order to show that T has a fixed point let $x_0 \in X_w$ be arbitrary and fixed. We define a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X_w$, $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$

If there exists $n_0 \in \mathbb{N}$ for which $x_{n0+1} = x_{n0}$, then $Tx_{n0} = x_{n0}$ and the proof is finished.

Now supposed that $x_{n+1} \neq x$ for all $n \in \mathbb{N}$. Then $w_\lambda(x_{n+1}, x_n) > 0$ for all $n \in \mathbb{N}$ and for all $\lambda > 0$, using (1), the following holds for all $n \in \mathbb{N}$ and for all $\lambda > 0$:

$$F(w_\lambda(x_{n+1}, x_n)) \leq F(w_\lambda(x_n, x_{n-1})) - \tau \qquad \leq F(w_\lambda(x_{n-1}, x_{n-2})) - 2\tau$$

$$\ldots$$

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$$\leq F\,(w_\lambda(x_1, x_0)) - n\tau. \tag{2}$$

From (2), we get $\lim_{n\to\infty} F(w_\lambda(x_{n+1}, x_n)) = -\infty$ that together with (F2) gives

$$\lim_{n\to\infty} F(w_\lambda(x_{n+1}, x_n)) = 0,$$

for all $\lambda > 0$. So for each $\lambda > 0$, we have for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $w_\lambda(x_{n+1}, x_n) < \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Without loss of generality, suppose $m, n \in \mathbb{N}$ and $m > n$. Observe that, for $\frac{\lambda}{m-n} > 0$ there exists $n_{\lambda/(m-n)} \in \mathbb{N}$ such that

$$w_{\frac{\lambda}{m-n}}(x_{n+1}, x_n) < \frac{\varepsilon}{m-n},$$

for all $n \geq n_{\lambda/(m-n)}$. Now we have

$$\begin{aligned} w_\lambda(x_n, x_m) &\leq w_{\frac{\lambda}{m-n}}(x_{n+1}, x_n) + w_{\frac{\lambda}{m-n}}(x_{n+1}, x_{n+2}) + \cdots + w_{\frac{\lambda}{m-n}}(x_{m-1}, x_m) \\ &< \frac{\varepsilon}{m-n} + \frac{\varepsilon}{m-n} + \cdots + \frac{\varepsilon}{m-n} \\ &= \varepsilon \end{aligned}$$

for all $m,\ n \geq n_{\lambda/(m-n)} \in \mathrm{N}$. This implies $\{x_n\}_{n\in\mathrm{N}}$ is a Cauchy sequence.

From the completeness of X_w there exists $x^* \in X_w$ such that $\lim_{n\to\infty} x_n = x.^*$

Finally, from (F1) and (1) we get that

$$w_\lambda(Tx, Ty) < w_\lambda(x,y),$$

for all $x, y \in \mathrm{X}_w$, $Tx \neq Ty$ and $\lambda > 0$. Thus

$$\begin{aligned} w_\lambda(Tx^*, x^*) &= \lim_{n\to\infty} w_\lambda(Tx^*, x_n) \\ &= \lim_{n\to\infty} w_\lambda(Tx^*, Tx_{n-1}) \\ &\leq \lim_{n\to\infty} w_\lambda(x^*, x_{n-1}) = 0, \end{aligned}$$

for all $\lambda > 0$, which completes the proof.

Example 3.3. Let $X = \{(a,0) \in R^2 | a \geq 0\} \cup \{(0,b) \in R^2 | b > 0\}$. Defined the mapping $w : (0,\infty) \times X \times X \rightarrow [0, \infty)$ by

$$w_\lambda((a_1, 0),(a_2, 0))= \frac{|a_1 - a_2|}{\lambda},$$

$$w_\lambda((0, b_1), (0, b_2))= \frac{|b_1 - b_2|}{3\lambda},$$

and

$$w_\lambda((a, 0),(0, b))= \frac{a}{\lambda} + \frac{b}{3\lambda} = w_\lambda((0, b),(a, 0))$$

We note that if we take $\lambda \rightarrow \infty$, then we see that $X = X_w$ and also T and F is define by

$$T((a, 0)) = (0, 2a),$$

$$T((0, b))= (\frac{b}{6}, 0),$$

and

$$F(\alpha) = \ln \alpha.$$

We can imply that $w_\lambda(Tx, Ty) \leq \frac{2}{3} w_\lambda(x, y)$ for all $x, y \in X$ and all $\lambda > 0$.

Indeed, case1. let $x = (a_1, 0)$, $y = (a_2, 0)$, then

$$w_\lambda(Tx, Ty) = w_\lambda(T(a_1, 0), T(a_2, 0)) = w_\lambda((0, 2a_1), (0, 2a_2))$$

$$= \frac{2|a1-a2|}{3\lambda},$$

$$w_\lambda(x,y) = w_\lambda((a_1, 0),(a_2, 0)) = \frac{|a_1-a_2|}{\lambda},$$

$$w_\lambda(T_x, T_x) = \frac{2}{3} w_\lambda(x,y).$$

Case2. let $x = (0, b_1)$, $y = (0, b_2)$, we have

$$w_\lambda(Tx, Ty) = w_\lambda(T(0, b_1), (0, b_2)) = w_\lambda((\frac{b_1}{6},0),(\frac{b_2}{6},0))$$

$$= \frac{|b_1-b_2|}{6\lambda},$$

$$w_\lambda(x, y) = w_\lambda((0, b_1), (0, b_2)) = \frac{|b_1-b_2|}{3\lambda},$$

$$w_\lambda(Tx, Ty) = \frac{1}{2} w_\lambda(x, y).$$

Case3. let $x = (a, 0)$, $y = (0, b)$, then

$$w_\lambda(Tx, Ty) = w_\lambda(T(a, 0), T(0, b)) = w_\lambda((0, 2a), (\frac{b}{6}, 0))$$

$$= \frac{2a}{3\lambda} + \frac{b}{6\lambda},$$

$$w_\lambda(x, y) = w_\lambda((a, 0),(0, b)) = \frac{a}{\lambda} + \frac{b}{3\lambda},$$

$$w_\lambda(Tx, Ty) \leq \frac{2}{3} w_\lambda(x, y).$$

Hence we have

$$w_\lambda(Tx, Ty) \leq \frac{2}{3} w_\lambda(x, y).$$

for all $\lambda > 0$ and $x,y \in X$. *Let* $\tau = \ln\frac{3}{2}$, we can get that

$$e^{-\tau} = \frac{2}{3},$$

$$w_\lambda(Tx, Ty) \leq e^{-\tau} w_\lambda(x, y),$$

$$\tau + \ln(w_\lambda(Tx, Ty)) \leq \ln(w_\lambda(x, y)),$$

$$\tau + F(w_\lambda(Tx, Ty)) \leq F(w_\lambda(x, y)),$$

for all $\lambda > 0$, all $x, y \in X$ and $Tx \neq Ty$. Thus T is a F -contractive mapping. Therefore, T has a unique fixed point that is $(0, 0) \in X_w$.

On the Euclidean metric d on X_w, we see that

$$d\left(T(1,0), T\left(0, \frac{1}{2}\right)\right) = d\left((0,2), \left(\frac{1}{12}\right)\right) > d\left((1,0), (0, \frac{1}{2})\right)$$

Thus, T is not a F -contraction on standard metric space.

[1] Banach, S.: Sur les opérations dans les ensembles abtraits et leur applications aux équations intégrales. Fundam. Math. 3, 133-181(1992).

[2] Gregori, V., Sapena, A.: On fixed point theorems in fuzzy metric spaces. Fuzzy Sets Syst. 125, 245-252(2002).

[3] Du, W. S.: A note on cone metric spaces and its equivalence. Nonlinear Anal. 72, 2259-2261(2010).

[4] Azam, A., Fssher, B., Khan, M.: Commen fixed points theorems in complete valued metric spaces. Nonlinear Anal. Optim. 32(3), 243-253(2011).

[5] Altun, I., Minak, G., Dag, H.: Multivalued F -contractions on complete metric spaces. J. Nonlinear Convex Anal. 16(4), 659-666(2015).

[6] Wardowski, B.: Fixed point theory of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. 2012. (94)2012.

[7] Chistyakov, W.: Modular metric spaces, I: basic concepts.Nonlinear Anal. 72, 1-14(2010).

[8] Chistyakov, W.: Modular metric spaces generated by F-modular. Folia Math. 14, 3-25(2008).

[9] Chistyakov, W.: Metric Modular spaces and their application.Dokl Math. 73(1)32-35(2006).

[10] Padcharoen, A., Gopal, D., Chaipunya, P., Kumam, P.: Fixed point and periodic point results for α-type F-contractions in modular metric spaces. Fixed Point Theory Appl. 2016. (39)2016.