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Construction of Unextendible Maximally Entangled Bases in Specific Space

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Abstract: We study the construction of unextendible maximally entangled bases (UMEB) from lower space to the higher space. Then we give the concrete forms of the UMEBs in $C^{10} \otimes C^{12}$, $C^{10} \otimes C^{15}$ and $C^{10} \otimes C^{18}$ from the UMEBs in $C^5 \otimes C^5$ and $C^5 \otimes C^6$.

Keywords: Maximally entangled bases (MEB); Unextendible maximally entangled bases (UMEB); Bipartite space

1 Introduction

Quantum entanglement is a fundamental feature of quantum mechanics and has been extensively investigated in recent years[1]. The importance of quantum entanglement has been demonstrated in various applications such as cryptographic protocols, quantum error correction codes and quantum state tomography[2]. One of the important problem in this filed is to characterize entanglement mathematically. The concept of unextendible product basis (UPB)[3] is useful in studying entanglement, it is a set of incomplete orthonormal product basis whose complementary space has no product states.

As a generalization of UPB, the notion of unextendible maximally entangled basis (UMEB)[4] has been proposed: a set of orthonormal maximally entangled states in $C^d \otimes C^{d'}$ consisting of fewer than d^2 vectors which have no additional maximally entangled vectors orthogonal to all of them. The definition of UMEB is as follows: A set of pure states $\{|\Psi_a\rangle\}_{i=0}^{n-1} \in C^d \otimes C^{d'}$, called an unextendible maximally entangled bases (UMEB)[1] if and only if (i) All states $\{|\Psi_a\rangle\}_{i=0}^{n-1}$ are maximally entangled ; (ii) $\langle\Psi_a|\Psi_b\rangle = \delta_{a,b}$, $a,b \in 0,\cdots,n-1$; (iii) If $\langle\Psi_a|\Psi\rangle = 0$ for all $a = 0,\cdots,n-1$ then $|\Psi\rangle$ cannot be maximally entangled.

Studying about the construction of the UMEB has been increase recently, Wang et al. [5] and Guo

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et al.[6] put forward a method of constructing UMEBs in $C^{qd}\otimes C^{qd}$ from that in $C^{d}\otimes C^{d}$. Zhang et al. constructed a $pqd^2-p(d^2-N)-$ member UMEB in $C^{pd}\otimes C^{qd}\,(p\le q)$ from $C^{d}\otimes C^{d}$ or $C^{p}\otimes C^{q}$ in Ref. [7]. Later, Shi et al.[8] generalized the above constructions and proposed a $(pqdd'-p(dd'-N))-$member UMEB in $C^{pd}\otimes C^{qd'}\,(p\le q)$ from $C^{d}\otimes C^{d'}$.

In this paper, we study the construction of the UMEB in lower space to higher space and give three examples to illustrate it. We start with the construction of a 146-member UMEB in $C^{10}\otimes C^{15}$ from a 23-member UMEB in $C^{5}\otimes C^{5}$, then we show a 110-member UMEB in $C^{10}\otimes C^{12}$ from a 25-member UMEB in $C^{5}\otimes C^{6}$, at last, we give a 170-member UMEB in $C^{10}\otimes C^{18}$ from a 25-member UMEB in $C^{5}\otimes C^{6}$.

2 UMEB in $C^{10}\otimes C^{15}$ from $C^{5}\otimes C^{5}$

Based on the general construction of the UMEBs in $C^{pd}\otimes C^{qd}\,(p\le q)$ from the UMEBs in $C^{d}\otimes C^{d}$ proposed by lemma 1, we can construct a 146-member UMEB in $C^{10}\otimes C^{15}$ from a 23-member UMEB in $C^{5}\otimes C^{5}$.

Lemma 1[7]**.** If there is an $N-$member UMEB $\{|\Psi_j\rangle\}$ in $C^{d}\otimes C^{d}$, then there exists a $pqd^2-p(d^2-N)-$member UMEB in $C^{pd}\otimes C^{qd}\,(p\le q)$.

Let $\{W_j=[w^j_{i'i}]\}_{j=0}^{N-1}$ be a UUB of $M_{d\times d}$ corresponding to $\{|\Psi_j\rangle\}$,

$$|\Psi_j\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\sum_{i=0}^{d-1}w^j_{i'i}|i\rangle\otimes|i'\rangle,\ j=0,\cdots,N-1.$$

Denote

$$U_{nm}=\sum_{a=0}^{d-1}e^{\frac{2\pi na\sqrt{-1}}{d}}|a\oplus_d m\rangle\langle a|,$$

$$V_{kl}=\sum_{a=0}^{p-1}e^{\frac{2\pi ka\sqrt{-1}}{d}}|a\oplus_q l\rangle\langle a|,$$

where $m,n=0,\cdots,d-1;\quad l=0,\cdots,q-1;\quad j=0,\cdots,N-1,$ and

$$B^j_{k0}=V_{k0}\otimes W_j,\quad k=0,\cdots,p-1;\quad j=0,\cdots,N-1,$$

$$B^{nm}_{kl}=V_{kl}\otimes U_{nm},\quad k=0,\cdots,p-1;\quad l=1,\cdots q-1;\quad m,n=0,\cdots,d-1.$$

Set $C_1=\{B^j_{k0}\}$ and $C_2=\{B^{nm}_{kl}\}$. Then, $C_1\cup C_2$ is exactly a USV1B in $M_{qd\times pd}$.

Next we will give a specific example of UMEB in $C^{10}\otimes C^{15}$ to illustrate lemma 1.

Proposition 1. There is an $23-$ member UMEB $\{|\Psi_j\rangle\}$ in $C^5 \otimes C^5$, then there exists a $146-$member UMEB in $C^{10} \otimes C^{15}$.

A 23-member UMEB in $C^5 \otimes C^5$ from Ref. [9] is as follows:

$$
\begin{aligned}
&|\Phi_{0\text{-}4}\rangle = 1/\sqrt{5}(|01'\rangle + \alpha|12'\rangle + \alpha^2|23'\rangle + \alpha^3|34'\rangle + \alpha^4|40'\rangle),\\
&|\Phi_{5\text{-}9}\rangle = 1/\sqrt{5}(|02'\rangle + \alpha|13'\rangle + \alpha^2|24'\rangle + \alpha^3|30'\rangle + \alpha^4|41'\rangle),\\
&|\Phi_{10\text{-}14}\rangle = 1/\sqrt{5}(|01'\rangle + \alpha|12'\rangle + \alpha^2|23'\rangle + \alpha^3|34'\rangle + \alpha^4|40'\rangle),\\
&|\Phi_{15\text{-}19}\rangle = 1/\sqrt{5}(|04'\rangle + \alpha|10'\rangle + \alpha^2|21'\rangle + \alpha^3|32'\rangle + \alpha^4|43'\rangle),\\
&|\Phi_{20}\rangle = 1/\sqrt{5}(|00'\rangle + |11'\rangle + |22'\rangle + |33'\rangle + |44'\rangle),\\
&|\Phi_{21}\rangle = 1/\sqrt{5}(|00'\rangle - |11'\rangle + |22'\rangle + \omega|33'\rangle + \omega^2|44'\rangle),\\
&|\Phi_{22}\rangle = 1/\sqrt{5}(\beta_1|00'\rangle + \beta_2|11'\rangle + \beta_3|22'\rangle + \beta_4|33'\rangle + \beta_5|44'\rangle).
\end{aligned}
$$

where $\alpha = 1, \omega, \omega^2, \omega^3, \omega^4$ and $\beta_1 = \sqrt{\frac{5}{6}} + \sqrt{\frac{1}{6}}i,\ \beta_2 = i,\ \beta_3 = -\sqrt{\frac{5}{6}} + \sqrt{\frac{1}{6}}i,\ \beta_4 = \frac{\sqrt{6}\omega^2 - \omega - 1}{3}i,\ \beta_5 = \frac{\sqrt{6}\omega - \omega^2 - 1}{3}i.$

Then, denote

$$
U_{nm} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^m \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^3 & 0 \\ 0 & 0 & 0 & 0 & \omega^4 \end{pmatrix}^n, \quad m, n = 0, \cdots, 4,
$$

$$
V_{kl} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^l \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^k, \quad k = 0, 1; \quad l = 0, 1, 2,
$$

where $\omega_3 = e^{\frac{2\pi\sqrt{-1}}{3}}$.

Let

$$
B_{01}^{nm} = \begin{pmatrix} 0 & 0 \\ U_{nm} & 0 \\ 0 & U_{nm} \end{pmatrix}, \quad B_{11}^{nm} = \begin{pmatrix} 0 & 0 \\ U_{nm} & 0 \\ 0 & -U_{nm} \end{pmatrix},
$$

$$
B_{02}^{nm} = \begin{pmatrix} 0 & U_{nm} \\ 0 & 0 \\ U_{nm} & 0 \end{pmatrix}, \quad B_{12}^{nm} = \begin{pmatrix} 0 & U_{nm} \\ 0 & 0 \\ -U_{nm} & 0 \end{pmatrix},
$$

$$
B_{00}^{j} = \begin{pmatrix} W_j & 0 \\ 0 & W_j \\ 0 & 0 \end{pmatrix}, \quad B_{10}^{j} = \begin{pmatrix} W_j & 0 \\ 0 & -W_j \\ 0 & 0 \end{pmatrix},
$$

where $n, m = 0, \cdots, 4;\ \mathrm{j} = 0, \cdots, 22.$

According to the substitution calculation of the above structure, we can get the following specific expression of the UMEB in $C^{10}\otimes C^{15}$.

$$
\begin{aligned}
&|\phi_{0-9}\rangle=1/\sqrt{5}\big[|01'\rangle+\alpha|12'\rangle+\alpha^2|23'\rangle+\alpha^3|34'\rangle+\alpha^4|40'\rangle\pm(|56'\rangle+\alpha|67'\rangle+\alpha^2|78'\rangle+\alpha^3|89'\rangle+\alpha^4|95'\rangle)\big],\\
&|\phi_{10-19}\rangle=1/\sqrt{5}\big[|02'\rangle+\alpha|13'\rangle+\alpha^2|24'\rangle+\alpha^3|30'\rangle+\alpha^4|41'\rangle\pm(|57'\rangle+\alpha|68'\rangle+\alpha^2|79'\rangle+\alpha^3|85'\rangle+\alpha^4|96'\rangle)\big],\\
&|\phi_{20-29}\rangle=1/\sqrt{5}\big[|03'\rangle+\alpha|14'\rangle+\alpha^2|20'\rangle+\alpha^3|31'\rangle+\alpha^4|42'\rangle\pm(|58'\rangle+\alpha|69'\rangle+\alpha^2|75'\rangle+\alpha^3|86'\rangle+\alpha^4|97'\rangle)\big],\\
&|\phi_{30-39}\rangle=1/\sqrt{5}\big[|04'\rangle+\alpha|10'\rangle+\alpha^2|21'\rangle+\alpha^3|33'\rangle+\alpha^4|43'\rangle\pm(|59'\rangle+\alpha|65'\rangle+\alpha^2|76'\rangle+\alpha^3|87'\rangle+\alpha^4|98'\rangle)\big],\\
&|\phi_{40-49}\rangle=1/\sqrt{5}\big[|05'\rangle+\alpha|16'\rangle+\alpha^2|27'\rangle+\alpha^3|38'\rangle+\alpha^4|49'\rangle\pm(|5,10'\rangle+\alpha|6,11'\rangle+\alpha^2|7,12'\rangle+\alpha^3|8,13'\rangle+\alpha^4|9,14'\rangle)\big],\\
&|\phi_{50-59}\rangle=1/\sqrt{5}\big[|06'\rangle+\alpha|17'\rangle+\alpha^2|28'\rangle+\alpha^3|39'\rangle+\alpha^4|45'\rangle\pm(|5,11'\rangle+\alpha|6,12'\rangle+\alpha^2|7,13'\rangle+\alpha^3|8,14'\rangle+\alpha^4|9,15'\rangle)\big],\\
&|\phi_{60-69}\rangle=1/\sqrt{5}\big[|07'\rangle+\alpha|18'\rangle+\alpha^2|29'\rangle+\alpha^3|35'\rangle+\alpha^4|46'\rangle\pm(|5,12'\rangle+\alpha|6,13'\rangle+\alpha^2|7,14'\rangle+\alpha^3|8,10'\rangle+\alpha^4|9,11'\rangle)\big],\\
&|\phi_{70-79}\rangle=1/\sqrt{5}\big[|08'\rangle+\alpha|19'\rangle+\alpha^2|25'\rangle+\alpha^3|36'\rangle+\alpha^4|47'\rangle\pm(|5,13'\rangle+\alpha|6,14'\rangle+\alpha^2|7,10'\rangle+\alpha^3|8,11'\rangle+\alpha^4|9,12'\rangle)\big],\\
&|\phi_{80-89}\rangle=1/\sqrt{5}\big[|09'\rangle+\alpha|15'\rangle+\alpha^2|26'\rangle+\alpha^3|37'\rangle+\alpha^4|48'\rangle\pm(|5,14'\rangle+\alpha|6,10'\rangle+\alpha^2|7,11'\rangle+\alpha^3|8,12'\rangle+\alpha^4|9,13'\rangle)\big],\\
&|\phi_{90-99}\rangle=1/\sqrt{5}\big[|0,10'\rangle+\alpha|1,11'\rangle+\alpha^2|2,12'\rangle+\alpha^3|3,13'\rangle+\alpha^4|4,14'\rangle\pm(|50'\rangle+\alpha|61'\rangle+\alpha^2|72'\rangle+\alpha^3|83'\rangle+\alpha^4|94'\rangle)\big],\\
&|\phi_{100-109}\rangle=1/\sqrt{5}\big[|0,11'\rangle+\alpha|1,12'\rangle+\alpha^2|2,13'\rangle+\alpha^3|3,14'\rangle+\alpha^4|4,10'\rangle\pm(|51'\rangle+\alpha|62'\rangle+\alpha^2|73'\rangle+\alpha^3|84'\rangle+\alpha^4|90'\rangle)\big],\\
&|\phi_{110-119}\rangle=1/\sqrt{5}\big[|0,12'\rangle+\alpha|1,13'\rangle+\alpha^2|2,14'\rangle+\alpha^3|3,10'\rangle+\alpha^4|4,11'\rangle\pm(|52'\rangle+\alpha|63'\rangle+\alpha^2|74'\rangle+\alpha^3|80'\rangle+\alpha^4|91'\rangle)\big],\\
&|\phi_{120-129}\rangle=1/\sqrt{5}\big[|0,13'\rangle+\alpha|1,14'\rangle+\alpha^2|2,10'\rangle+\alpha^3|3,11'\rangle+\alpha^4|4,12'\rangle\pm(|53'\rangle+\alpha|64'\rangle+\alpha^2|70'\rangle+\alpha^3|81'\rangle+\alpha^4|92'\rangle)\big],\\
&|\phi_{130-139}\rangle=1/\sqrt{5}\big[|0,14'\rangle+\alpha|1,10'\rangle+\alpha^2|2,11'\rangle+\alpha^3|3,12'\rangle+\alpha^4|4,13'\rangle\pm(|54'\rangle+\alpha|60'\rangle+\alpha^2|71'\rangle+\alpha^3|82'\rangle+\alpha^4|93'\rangle)\big],\\
&|\phi_{140,141}\rangle=1/\sqrt{5}\big[|00'\rangle+|11'\rangle+|22'\rangle+|33'\rangle+|44'\rangle\pm(|55'\rangle+|66'\rangle+|77'\rangle+|88'\rangle+|99'\rangle)\big],\\
&|\phi_{142,143}\rangle=1/\sqrt{5}\big[|00'\rangle-|11'\rangle+|22'\rangle+\omega|33'\rangle+\omega^2|44'\rangle\pm(|55'\rangle-|66'\rangle+|77'\rangle+\omega|88'\rangle+\omega^2|99'\rangle)\big],\\
&|\phi_{144,145}\rangle=1/\sqrt{5}\big[\beta_1|00'\rangle+\beta_2|11'\rangle+\beta_3|22'\rangle+\beta_4|33'\rangle+\beta_5|44'\rangle\pm(\beta_1|55'\rangle+\beta_2|66'\rangle+\beta_3|77'\rangle+\beta_4|88'\rangle+\beta_5|99'\rangle)\big].
\end{aligned}
$$

where $\alpha=1,\omega,\omega^2,\omega^3,\omega^4$ and $\beta_1=\sqrt{\frac{5}{6}}+\sqrt{\frac{1}{6}}i,\ \beta_2=i,\ \beta_3=-\sqrt{\frac{5}{6}}+\sqrt{\frac{1}{6}}i,\ \beta_4=\frac{\sqrt{6}\omega^2-\omega-1}{3}i,\ \beta_5=\frac{\sqrt{6}\omega-\omega^2-1}{3}i.$

3 UMEB in $C^{10}\otimes C^{12}$ from $C^5\otimes C^6$

In this part, we mainly discuss the construction of the UMEB in $C^{pd}\otimes C^{qd}\,(p\le q)$ from that in $C^p\otimes C^q$, and then we give a concrete example to show it.

Lemma 2[7]. If there is an $N-$member UMEB $\{|\Psi_j\rangle\}$ in $C^p\otimes C^q$, then there exists a $pqd^2-d(\mathrm{pq}-N)-$member UMEB in $C^{pd}\otimes C^{qd}\,(p\le q)$.

Let $\{W_j=[w^j_{i'i}]\}_{j=0}^{N-1}$ be a USV1B of $M_{q\times p}$ corresponding to $\{|\Psi_j\rangle\}$,

$$|\Psi_j\rangle=\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}\sum_{i=0}^{d-1}w^j_{i'i}|i\rangle\otimes|i'\rangle,\ j=0,\cdots,N-1.$$

Denote

$$U_{nm}=\sum_{a=0}^{d-1}e^{\frac{2\pi na\sqrt{-1}}{d}}|a\oplus_d m\rangle\langle a|,$$

$$V_{kl}=\sum_{a=0}^{p-1}e^{\frac{2\pi ka\sqrt{-1}}{d}}|a\oplus_q l\rangle\langle a|,$$

where $m,n=0,\cdots,d-1;\quad l=0,\cdots,q-1;\quad j=0,\cdots,N-1,$ and

$$B_{n0}^{j}=U_{n0}\otimes W_j,\quad n=0,\cdots,d-1;\quad j=0,\cdots,N-1,$$

$$B_{nm}^{kl}=U_{nm}\otimes V_{kl},\quad m=1,\cdots,d-1;\quad n=0,\cdots,d-1;\quad m=1,\cdots,q-1;\quad k=1,\cdots,p-1,$$

Set $C_1=\{B_{n0}^{j}\}$ and $C_2=\{B_{nm}^{kl}\}$. Then, $C_1\bigcup C_2$ is exactly a USV1B in $M_{qd\times pd}$.

Next we will begin to construct the UMEB in $C^{10}\otimes C^{12}$ from the UMEB in $C^5\otimes C^6$.

Proposition 2. There is an $25-$member UMEB $\{|\Psi_j\rangle\}$ in $C^5\otimes C^6$, then there exists a $110-$member UMEB in $C^{10}\otimes C^{12}$.

First, we can get a 25-member UMEB in $C^5\otimes C^6$ from Ref. [10]:

$$\begin{aligned}
|\Phi_{0\text{-}4}\rangle&=1/\sqrt{5}(|01'\rangle+\alpha|25'\rangle+\alpha^2|40'\rangle+\alpha^3|12'\rangle+\alpha^4|34'\rangle),\\
|\Phi_{5\text{-}9}\rangle&=1/\sqrt{5}(|05'\rangle+\alpha|20'\rangle+\alpha^2|42'\rangle+\alpha^3|14'\rangle+\alpha^4|33'\rangle),\\
|\Phi_{10\text{-}14}\rangle&=1/\sqrt{5}(|00'\rangle+\alpha|22'\rangle+\alpha^2|44'\rangle+\alpha^3|13'\rangle+\alpha^4|31'\rangle),\\
|\Phi_{15\text{-}19}\rangle&=1/\sqrt{5}(|02'\rangle+\alpha|24'\rangle+\alpha^2|41'\rangle+\alpha^3|15'\rangle+\alpha^4|30'\rangle),\\
|\Phi_{20\text{-}24}\rangle&=1/\sqrt{5}(|04'\rangle+\alpha|21'\rangle+\alpha^2|45'\rangle+\alpha^3|10'\rangle+\alpha^4|32'\rangle).
\end{aligned}$$

where $\alpha=1,\omega,\omega^2,\omega^3,\omega^4$.

Denote

$$U_{nm}=\begin{pmatrix}0&1\\1&0\end{pmatrix}^m\cdot\begin{pmatrix}1&0\\0&-1\end{pmatrix}^n,\quad m,n=0,1,$$

$$V_{kl}=\begin{pmatrix}0&0&0&0&0&1\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{pmatrix}^l\cdot\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0&0\\0&\omega&0&0&0\\0&0&\omega^2&0&0\\0&0&0&\omega^3&0\\0&0&0&0&\omega^4\end{pmatrix}^k,\quad k=0,\cdots,4;\quad l=0,\cdots5,$$

where $\omega_3=e^{\frac{2\pi\sqrt{-1}}{3}}$.

Let

$$B_{01}^{kl}=\begin{pmatrix}0 & V_{kl}\\ V_{kl} & 0\end{pmatrix},\quad B_{11}^{kl}=\begin{pmatrix}0 & V_{kl}\\ -V_{kl} & 0\end{pmatrix},$$
$$B_{00}^{j}=\begin{pmatrix}W_j & 0\\ 0 & W_j\end{pmatrix},\quad B_{10}^{j}=\begin{pmatrix}W_j & 0\\ 0 & -W_j\end{pmatrix},$$

where $k,l=0,1;\ j=0,\cdots,24.$

Then, we can get the following $110-$UMEB in $C^{10}\otimes C^{12}$:

$$|\phi_{0-9}\rangle=1/\sqrt{5}\left[|01'\rangle+\alpha|25'\rangle+\alpha^2|40'\rangle+\alpha^3|12'\rangle+\alpha^4|34'\rangle\pm\left(|57'\rangle+\alpha|7,11'\rangle+\alpha^2|96'\rangle+\alpha^3|68'\rangle+\alpha^4|8,10'\rangle\right)\right],$$
$$|\phi_{10-19}\rangle=1/\sqrt{5}\left[|05'\rangle+\alpha|20'\rangle+\alpha^2|42'\rangle+\alpha^3|14'\rangle+\alpha^4|33'\rangle\pm\left(|5,11'\rangle+\alpha|76'\rangle+\alpha^2|98'\rangle+\alpha^3|6,10'\rangle+\alpha^4|89'\rangle\right)\right],$$
$$|\phi_{20-29}\rangle=1/\sqrt{5}\left[|00'\rangle+\alpha|22'\rangle+\alpha^2|44'\rangle+\alpha^3|13'\rangle+\alpha^4|31'\rangle\pm\left(|56'\rangle+\alpha|78'\rangle+\alpha^2|9,10'\rangle+\alpha^3|69'\rangle+\alpha^4|87'\rangle\right)\right],$$
$$|\phi_{30-39}\rangle=1/\sqrt{5}\left[|02'\rangle+\alpha|24'\rangle+\alpha^2|41'\rangle+\alpha^3|15'\rangle+\alpha^4|30'\rangle\pm\left(|58'\rangle+\alpha|7,10'\rangle+\alpha^2|97'\rangle+\alpha^3|6,11'\rangle+\alpha^4|86'\rangle\right)\right],$$
$$|\phi_{40-49}\rangle=1/\sqrt{5}\left[|04'\rangle+\alpha|21'\rangle+\alpha^2|45'\rangle+\alpha^3|10'\rangle+\alpha^4|32'\rangle\pm\left(|5,10'\rangle+\alpha|77'\rangle+\alpha^2|9,11'\rangle+\alpha^3|66'\rangle+\alpha^4|88'\rangle\right)\right],$$
$$|\phi_{50-59}\rangle=1/\sqrt{5}\left[|06'\rangle+\alpha|17'\rangle+\alpha^2|28'\rangle+\alpha^3|3,9'\rangle+\alpha^4|4,10'\rangle\pm\left(|50'\rangle+\alpha|61'\rangle+\alpha^2|72'\rangle+\alpha^3|83'\rangle+\alpha^4|94'\rangle\right)\right],$$
$$|\phi_{60-69}\rangle=1/\sqrt{5}\left[|07'\rangle+\alpha|18'\rangle+\alpha^2|29'\rangle+\alpha^3|3,10'\rangle+\alpha^4|4,11'\rangle\pm\left(|51'\rangle+\alpha|62'\rangle+\alpha^2|73'\rangle+\alpha^3|84'\rangle+\alpha^4|95'\rangle\right)\right],$$
$$|\phi_{70-79}\rangle=1/\sqrt{5}\left[|08'\rangle+\alpha|19'\rangle+\alpha^2|2,10'\rangle+\alpha^3|3,11'\rangle+\alpha^4|4,6'\rangle\pm\left(|52'\rangle+\alpha|63'\rangle+\alpha^2|74'\rangle+\alpha^3|85'\rangle+\alpha^4|90'\rangle\right)\right],$$
$$|\phi_{80-89}\rangle=1/\sqrt{5}\left[|09'\rangle+\alpha|1,10'\rangle+\alpha^2|2,11'\rangle+\alpha^3|3,6'\rangle+\alpha^4|4,7'\rangle\pm\left(|53'\rangle+\alpha|64'\rangle+\alpha^2|75'\rangle+\alpha^3|80'\rangle+\alpha^4|91'\rangle\right)\right],$$
$$|\phi_{90-99}\rangle=1/\sqrt{5}\left[|0,10'\rangle+\alpha|1,11'\rangle+\alpha^2|2,6'\rangle+\alpha^3|3,7'\rangle+\alpha^4|4,8'\rangle\pm\left(|54'\rangle+\alpha|65'\rangle+\alpha^2|70'\rangle+\alpha^3|81'\rangle+\alpha^4|92'\rangle\right)\right],$$
$$|\phi_{100-109}\rangle=1/\sqrt{5}\left[|0,11'\rangle+\alpha|16'\rangle+\alpha^2|27'\rangle+\alpha^3|3,8'\rangle+\alpha^4|4,9'\rangle\pm\left(|55'\rangle+\alpha|60'\rangle+\alpha^2|71'\rangle+\alpha^3|82'\rangle+\alpha^4|93'\rangle\right)\right],$$

where $\alpha=1,\omega,\omega^2,\omega^3,\omega^4$.

3 UMEB in $C^{10}\otimes C^{18}$ from $C^5\otimes C^6$

We have introduced the construction of the UMEB in $C^{pd}\otimes C^{qd}\left(p\le q\right)$ from that in $C^d\otimes C^d$ and $C^p\otimes C^q$, UMEB in $C^{pd}\otimes C^{qd'}\left(p\le q\right)$ from the UMEB in $C^d\otimes C^{d'}$ will be naturally shown in the following part.

At the beginning of this part, we need to know prepare the following facts, let $M_{d\times d'}$ be the space of all $d\times d'$ complex matrices equipped with an inner product by $\left(A,B\right)=Tr\left(A^{\dagger}B\right)$. There is a one to one relation between the space $C^d\otimes C^{d'}$ and the space $M_{d\times d'}$ [11]:

$$\left|\Psi_i\right\rangle=\sum_{k,l}a_{kl}^{(i)}\left|k\right\rangle\left|l'\right\rangle\in C^d\otimes C^{d'}\Leftrightarrow A_i=\left(a_{kl}^{(i)}\right)\in M_{d\times d'},$$

$$\left\langle\Psi_i\middle|\Psi_j\right\rangle=Tr\left(A_i^{\dagger}A_j\right).$$

where $\left\{\left|k\right\rangle\right\}$ and $\left\{\left|l'\right\rangle\right\}$ are the standard computational bases of C^d and $C^{d'}$, respectively. A

$d\times d'$ matrix is called a singular -value-1 matrix if its nonzero singular values are $\{1,1,1,\cdots,1\}_d$.

Lemma 3[8]. If there is an $N-$ member UMEB in $C^d\otimes C^{d'}$, then there is a $(pqdd'-p(dd'-N))-$member UMEB in $C^{pd}\otimes C^{qd'}(p\le q)$.

Let $J_{p\times q}$ be a $p\times q$ matrix with all entries being 1, then $J_{p\times q}$ can be decomposed as $J_{p\times q}=P_0+P_1+\cdots+P_{q-1}$, where each P_i is a permutation matrix. For any $P_l=P_0T^l, l=0,\cdots q-1$, where

$$P_0=\begin{pmatrix}1&0&\cdots&0&0&\cdots&0\\0&1&\cdots&0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots&\vdots&&\vdots\\0&0&\cdots&1&0&\cdots&0\end{pmatrix}_{p\times q}\quad and\quad T=\begin{pmatrix}0&1&0&\cdots&0\\0&0&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&1\\1&0&0&\cdots&0\end{pmatrix}_{q\times q}$$

Define a $p\times q$ matrix $Q_l^a, a=0,\cdots,p-1,$ by

$$Q_l^a(i,j)=\begin{cases}0 & if\ P_l(i,j)=0,\\ \xi_p^{a(i-1)} & if\ P_l(i,j)=1,\end{cases}$$

where $\xi_p=e^{\frac{2\pi\sqrt{-1}}{p}}$. $M=(M_{i,j})_{p\times q}$ is a block matrix form belongs to $M_{pd\times qd'}$, L_l is a subspace of $M_{pd\times qd'}$ which consists of all block matrices M with $M_{ij}=0$ if $P_l(i,j)=0$. Then,

$$M_{pd\times qd'}=L_0\oplus L_1\oplus\cdots\oplus L_{q-1},$$

such that $\dim L_l=pdd'$ for all $l=0,\cdots,q-1$.

Let $\{A_j\}_{j=1}^N$ is an $N-$number USV1B in $M_{d\times d'}$, $\{B_{s,t}\}$ is an SV1B in $M_{d\times d'}$, where $s=0,\cdots d-1$, $t=0,\cdots,d'-1$. For any $a=0,\cdots,p-1$, $l=0,\cdots q-1$, $s=0,\cdots d-1$, $t=0,\cdots,d'-1$ and $j=1,\cdots N$, define

$$C_{a,l}^{s,t}=Q_l^a\otimes B_{s,t}\quad and\quad C_{a,0}^j=Q_l^a\otimes A_j.$$

Set $C_1=\{C_{a,l}^{s,t}\}$ and $C_2=\{C_{a0}^j\}$. Then, $C_1\cup C_2$ is exactly a USV1B in $M_{qd\times pd'}$.

Proposition 3. There is an $25-$member UMEB $\{|\Psi_j\rangle\}$ in $C^5\otimes C^6$, then there exists a $170-$member UMEB in $C^{10}\otimes C^{18}$.

Let $\{A_j\}_{j=1}^{25}$ be a $25-$member USV1B in $M_{5\times 6}$,

$$A_{1-5}=\begin{pmatrix}0&1&0&0&0&0\\0&0&\alpha^3&0&0&0\\0&0&0&0&0&\alpha\\0&0&0&0&\alpha^4&0\\\alpha^2&0&0&0&0&0\end{pmatrix},\quad A_{6-10}=\begin{pmatrix}0&0&0&0&0&1\\0&0&0&0&\alpha^3&0\\\alpha&0&0&0&0&0\\0&0&0&\alpha^4&0&0\\0&0&\alpha^2&0&0&0\end{pmatrix},$$

$$A_{11-15}=\begin{pmatrix}1&0&0&0&0&0\\0&0&0&\alpha^3&0&0\\0&0&\alpha&0&0&0\\0&\alpha^4&0&0&0&0\\0&0&0&0&\alpha^2&0\end{pmatrix},\quad A_{16-20}=\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&\alpha^3\\0&0&0&0&\alpha&0\\\alpha^4&0&0&0&0&0\\0&\alpha^2&0&0&0&0\end{pmatrix},$$

$$A_{21-25}=\begin{pmatrix}0&0&0&0&1&0\\\alpha^3&0&0&0&0&0\\0&\alpha&0&0&0&0\\0&0&\alpha^4&0&0&0\\0&0&0&0&0&\alpha^2\end{pmatrix}.$$

where $\alpha=1,\omega,\omega^2,\omega^3,\omega^4$.

Let

$$B_{s,t}=\begin{pmatrix}1&0&0&0&0&0\\0&(\omega)^s&0&0&0&0\\0&0&(\omega^2)^s&0&0&0\\0&0&0&(\omega^3)^s&0&0\\0&0&0&0&(\omega^4)^s&0\end{pmatrix}\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\1&0&0&0&0&0\end{pmatrix}^t,$$

where $s=0,\cdots,4,\quad t=0,\cdots,5.$ It is easy to see that $\{B_{s,t}\}$ is an SV1B in $M_{5\times6}$.

Let $J_{2\times3}=P_0+P_1+P_2,\ P_l=P_0T^l,\ l=0,1,2,$ where

$$P_0=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix},\quad T=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

For any $l=0,\cdots2,\ a=0,1,$ define a 2×3 matrix Q_l^a by

$$Q_l^a=\begin{cases}0 & if\ P_l(i,j)=0,\\(-1)^{a(i-1)} & if\ P_l(i,j)=1.\end{cases}$$

Then for each $s=0,\cdots,4,\quad t=0,\cdots,5$ and $j=1,\cdots,25,$

$$C_{0,1}^{s,t}=\begin{pmatrix}0 & B_{s,t} & 0\\ 0 & 0 & B_{s,t}\end{pmatrix},\quad C_{1,1}^{s,t}=\begin{pmatrix}0 & B_{s,t} & 0\\ 0 & 0 & -B_{s,t}\end{pmatrix},$$
$$C_{0,2}^{s,t}=\begin{pmatrix}0 & 0 & B_{s,t}\\ B_{s,t} & 0 & 0\end{pmatrix},\quad C_{1,2}^{s,t}=\begin{pmatrix}0 & 0 & B_{s,t}\\ -B_{s,t} & 0 & 0\end{pmatrix},$$
$$C_{0,0}^{j}=\begin{pmatrix}A_j & 0 & 0\\ 0 & A_j & 0\end{pmatrix},\quad C_{1,0}^{j}=\begin{pmatrix}A_j & 0 & 0\\ 0 & -A_j & 0\end{pmatrix},$$

Thus, a $170-$member UMEB in $C^{10}\otimes C^{18}$ can be described as follows:

$$|\phi_{0-9}\rangle=1/\sqrt{5}\left[|01'\rangle+\alpha|25'\rangle+\alpha^2|40'\rangle+\alpha^3|12'\rangle+\alpha^4|34'\rangle\pm\left(|57'\rangle+\alpha|7,11'\rangle+\alpha^2|96'\rangle+\alpha^3|68'\rangle+\alpha^4|8,10'\rangle\right)\right],$$
$$|\phi_{10-19}\rangle=1/\sqrt{5}\left[|05'\rangle+\alpha|20'\rangle+\alpha^2|42'\rangle+\alpha^3|14'\rangle+\alpha^4|33'\rangle\pm\left(|5,11'\rangle+\alpha|7,6'\rangle+\alpha^2|98'\rangle+\alpha^3|6,10'\rangle+\alpha^4|89'\rangle\right)\right],$$
$$|\phi_{20-29}\rangle=1/\sqrt{5}\left[|00'\rangle+\alpha|22'\rangle+\alpha^2|44'\rangle+\alpha^3|13'\rangle+\alpha^4|31'\rangle\pm\left(|56'\rangle+\alpha|7,8'\rangle+\alpha^2|9,10'\rangle+\alpha^3|69'\rangle+\alpha^4|87'\rangle\right)\right],$$
$$|\phi_{30-39}\rangle=1/\sqrt{5}\left[|02'\rangle+\alpha|24'\rangle+\alpha^2|41'\rangle+\alpha^3|15'\rangle+\alpha^4|30'\rangle\pm\left(|58'\rangle+\alpha|7,10'\rangle+\alpha^2|97'\rangle+\alpha^3|6,11'\rangle+\alpha^4|86'\rangle\right)\right],$$
$$|\phi_{40-49}\rangle=1/\sqrt{5}\left[|04'\rangle+\alpha|21'\rangle+\alpha^2|45'\rangle+\alpha^3|10'\rangle+\alpha^4|32'\rangle\pm\left(|5,10'\rangle+\alpha|7,7'\rangle+\alpha^2|9,11'\rangle+\alpha^3|66'\rangle+\alpha^4|88'\rangle\right)\right],$$
$$|\phi_{50-59}\rangle=1/\sqrt{5}\left[|06'\rangle+\alpha|17'\rangle+\alpha^2|28'\rangle+\alpha^3|3,9'\rangle+\alpha^4|4,10'\rangle\pm\left(|5,12'\rangle+\alpha|6,13'\rangle+\alpha^2|7,14'\rangle+\alpha^3|8,15'\rangle+\alpha^4|9,16'\rangle\right)\right],$$
$$|\phi_{60-69}\rangle=1/\sqrt{5}\left[|07'\rangle+\alpha|18'\rangle+\alpha^2|29'\rangle+\alpha^3|3,10'\rangle+\alpha^4|4,11'\rangle\pm\left(|5,13'\rangle+\alpha|6,14'\rangle+\alpha^2|7,15'\rangle+\alpha^3|8,16'\rangle+\alpha^4|9,17'\rangle\right)\right],$$
$$|\phi_{70-79}\rangle=1/\sqrt{5}\left[|08'\rangle+\alpha|19'\rangle+\alpha^2|2,10'\rangle+\alpha^3|3,11'\rangle+\alpha^4|4,6'\rangle\pm\left(|5,14'\rangle+\alpha|6,15'\rangle+\alpha^2|7,16'\rangle+\alpha^3|8,17'\rangle+\alpha^4|9,12'\rangle\right)\right],$$
$$|\phi_{80-89}\rangle=1/\sqrt{5}\left[|09'\rangle+\alpha|1,10'\rangle+\alpha^2|2,11'\rangle+\alpha^3|3,6'\rangle+\alpha^4|4,7'\rangle\pm\left(|5,15'\rangle+\alpha|6,16'\rangle+\alpha^2|7,17'\rangle+\alpha^3|8,12'\rangle+\alpha^4|9,13'\rangle\right)\right],$$
$$|\phi_{90-99}\rangle=1/\sqrt{5}\left[|0,10'\rangle+\alpha|1,11'\rangle+\alpha^2|2,6'\rangle+\alpha^3|3,7'\rangle+\alpha^4|4,8'\rangle\pm\left(|5,16'\rangle+\alpha|6,17'\rangle+\alpha^2|7,12'\rangle+\alpha^3|8,13'\rangle+\alpha^4|9,14'\rangle\right)\right],$$
$$|\phi_{100-109}\rangle=1/\sqrt{5}\left[|0,11'\rangle+\alpha|16'\rangle+\alpha^2|27'\rangle+\alpha^3|3,8'\rangle+\alpha^4|4,9'\rangle\pm\left(|5,17'\rangle+\alpha|6,12'\rangle+\alpha^2|7,13'\rangle+\alpha^3|8,14'\rangle+\alpha^4|9,15'\rangle\right)\right],$$
$$|\phi_{110-119}\rangle=1/\sqrt{5}\left[|0,12'\rangle+\alpha|1,13'\rangle+\alpha^2|2,14'\rangle+\alpha^3|3,15'\rangle+\alpha^4|4,16'\rangle\pm\left(|50'\rangle+\alpha|61'\rangle+\alpha^2|72'\rangle+\alpha^3|83'\rangle+\alpha^4|94'\rangle\right)\right],$$
$$|\phi_{120-129}\rangle=1/\sqrt{5}\left[|0,13'\rangle+\alpha|1,14'\rangle+\alpha^2|2,15'\rangle+\alpha^3|3,16'\rangle+\alpha^4|4,17'\rangle\pm\left(|51'\rangle+\alpha|62'\rangle+\alpha^2|73'\rangle+\alpha^3|84'\rangle+\alpha^4|95'\rangle\right)\right],$$
$$|\phi_{130-139}\rangle=1/\sqrt{5}\left[|0,14'\rangle+\alpha|1,15'\rangle+\alpha^2|2,16'\rangle+\alpha^3|3,17'\rangle+\alpha^4|4,12'\rangle\pm\left(|52'\rangle+\alpha|63'\rangle+\alpha^2|74'\rangle+\alpha^3|85'\rangle+\alpha^4|90'\rangle\right)\right],$$
$$|\phi_{140-149}\rangle=1/\sqrt{5}\left[|0,15'\rangle+\alpha|1,16'\rangle+\alpha^2|2,17'\rangle+\alpha^3|3,12'\rangle+\alpha^4|4,13'\rangle\pm\left(|53'\rangle+\alpha|64'\rangle+\alpha^2|75'\rangle+\alpha^3|80'\rangle+\alpha^4|91'\rangle\right)\right],$$
$$|\phi_{150-159}\rangle=1/\sqrt{5}\left[|0,16'\rangle+\alpha|1,17'\rangle+\alpha^2|2,12'\rangle+\alpha^3|3,13'\rangle+\alpha^4|4,14'\rangle\pm\left(|54'\rangle+\alpha|65'\rangle+\alpha^2|70'\rangle+\alpha^3|81'\rangle+\alpha^4|92'\rangle\right)\right],$$
$$|\phi_{160-169}\rangle=1/\sqrt{5}\left[|0,17'\rangle+\alpha|1,12'\rangle+\alpha^2|2,13'\rangle+\alpha^3|3,14'\rangle+\alpha^4|4,15'\rangle\pm\left(|55'\rangle+\alpha|60'\rangle+\alpha^2|71'\rangle+\alpha^3|82'\rangle+\alpha^4|93'\rangle\right)\right].$$

where $\alpha=1,\omega,\omega^2,\omega^3,\omega^4$.

4 Conclusion

We have listed three method about the construction of UMEB in a lower space to a higher space, which is helpful in the latter study of the UMEB. We have also presented three specific examples to illustrate the above three methods. It should be noted that the expression of the last method is slightly different from that of the other two methods, but both of them are effective ways to construct the UMEB.

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