APPLICATIONS OF GENERALIZED SCHMIDT DECOMPOSITION IN QUANTUM COHERENCE THEORY

Hongdan Yin, Yihao Sheng, Yuanhong Tao*#

Department of Mathematics, College of Sciences, Yanbian University,Yanji, Jilin, **CHINA**,133002

*Correspondence Author: Yuanhong Tao, E-mail: taoyuanhong12@126.com
#Supported by Natural Science Foundation of China under number 11761073.

ABSTRACT

Based on the standard form of generalized Schmidt decomposition of three qubit pure states, we discuss the super-additive relation, sub-additive relation, monogamy relation and other trade-off relations of quantum coherence measure. We first present correct proofs of super-additive relation and strong super-additive relation of l_1 norm coherence, then prove two sub-additive relations of l_1 norm coherence. We also present the conditions of the monogamy relations about l_1 norm coherence and the square of l_2 norm coherence separately, and finally establish their corresponding trade-off relations respectively.

Keywords: Generalized Schmidt decomposition; l_1 norm coherence; l_2 norm coherence; super-additive relation; sub-additive relation; monogamy relation.

I. Introduction

Generalized Schmidt decomposition aroused since there is no Schmidt decomposition for a general pure state of n partite quantum system. Aiming to reduce the number and the phase factor of coefficients that affect the state, A. Acin [1] proposed the concept of generalized Schmidt decomposition. He found that any three qubit pure state can be represented by one of the following three bases:

$$\{|000\rangle,|001\rangle,|010\rangle,|100\rangle,|111\rangle\};\ \{|000\rangle,|001\rangle,|110\rangle,|100\rangle,|111\rangle\};\ \{|000\rangle,|100\rangle,|110\rangle,|101\rangle,|111\rangle\}.$$

Later, Carteret et al. [2] extended the above conclusion to multipartite quantum states.

F. Liu et al. [3-4] first applied generalized Schmidt decomposition into quantum coherence theory. The quantum coherence comes from the principle of quantum superposition, which is the most basic property of quantum mechanics and the advantage of quantum information. Recent years, many coherence measures are proposed, such as l_1 norm coherence [5], l_2 norm coherence [6], relative entropy coherence [5], geometric coherence [7], skew information coherence [8] and so on.

According to the standard form of generalized Schmidt decomposition, this paper will discuss the super-additive relations, strong super-additive relations, sub-additive relations, monogamy relations and other trade-off relations for any qutrit pure state under l_1 norm coherence and the square of l_2 norm coherence.

II. Super-additive relation and strong super-additive relation

In this section, we discuss the applications of the generalized Schmidt decomposition of pure state in the super-additive relation and strong super-additive relation.

We first recall that the generalized Schmidt decomposition of pure state $|\psi\rangle_{ABC}$ in $2\otimes 2\otimes 2$ quantum system is [1]

$$|\psi\rangle_{ABC} = r_0|000\rangle + r_1 e^{i\theta}|100\rangle + r_2|101\rangle + r_3|110\rangle + r_4|111\rangle, \qquad (1.1)$$

where $r_i \geq 0$, $\sum_0^4 r_i^2 = 1$.

Let $\rho_{ABC} = |\psi\rangle_{ABC}\langle\psi|$, then we show the representation of ρ_{ABC} and its partial operators as follows:

$$\begin{aligned}\rho_{ABC} = {} & r_0^2|000\rangle\langle 000| + r_0 r_1 e^{-i\theta}|000\rangle\langle 100| + r_0 r_2|000\rangle\langle 101| + r_0 r_3|000\rangle\langle 110| + r_0 r_4|000\rangle\langle 111| \\ & + r_0 r_1 e^{i\theta}|100\rangle\langle 000| + r_1^2|100\rangle\langle 100| + r_1 r_2 e^{i\theta}|100\rangle\langle 101| + r_1 r_3 e^{i\theta}|100\rangle\langle 110| + r_1 r_4 e^{i\theta}|100\rangle\langle 111| \\ & + r_0 r_2|101\rangle\langle 000| + r_1 r_2 e^{-i\theta}|101\rangle\langle 100| + r_2^2|101\rangle\langle 101| + r_2 r_3|101\rangle\langle 110| + r_2 r_4|101\rangle\langle 111| \\ & + r_0 r_3|110\rangle\langle 000| + r_1 r_3 e^{-i\theta}|110\rangle\langle 100| + r_2 r_3|110\rangle\langle 101| + r_3^2|110\rangle\langle 110| + r_3 r_4|110\rangle\langle 111| \\ & + r_0 r_4|111\rangle\langle 000| + r_1 r_4 e^{-i\theta}|111\rangle\langle 100| + r_2 r_4|111\rangle\langle 101| + r_3 r_4|111\rangle\langle 110| + r_4^2|111\rangle\langle 111|,\end{aligned}$$

$$\begin{aligned}\rho_{AB} = tr_C(\rho_{ABC}) = {} & r_0^2|00\rangle\langle 00| + r_0 r_1 e^{-i\theta}|00\rangle\langle 10| + r_0 r_3|00\rangle\langle 11| + r_0 r_1 e^{i\theta}|10\rangle\langle 00| + (r_1^2 + r_2^2)|10\rangle\langle 10| \\ & + (r_1 r_3 e^{i\theta} + r_2 r_4)|10\rangle\langle 11| + r_0 r_3|11\rangle\langle 00| + (r_1 r_3 e^{-i\theta} + r_2 r_4)|11\rangle\langle 10| + (r_3^2 + r_4^2)|11\rangle\langle 11|,\end{aligned}$$

$$\rho_{AC}=tr_B(\rho_{ABC})=r_0^2|00\rangle\langle 00|+r_0r_1e^{-i\theta}|00\rangle\langle 10|+r_0r_2|00\rangle\langle 11|+r_0r_1e^{i\theta}|10\rangle\langle 00|+(r_1^2+r_3^2)|10\rangle\langle 10|$$
$$+(r_1r_2e^{i\theta}+r_3r_4)|10\rangle\langle 11|+r_0r_2|11\rangle\langle 00|+(r_1r_2e^{-i\theta}+r_3r_4)|11\rangle\langle 10|+(r_2^2+r_4^2)|11\rangle\langle 11|,$$

$$\rho_{BC}=tr_A(\rho_{ABC})=(r_0^2+r_1^2)|00\rangle\langle 00|+r_1r_2e^{i\theta}|00\rangle\langle 01|+r_1r_3e^{i\theta}|00\rangle\langle 10|+r_1r_4e^{i\theta}|00\rangle\langle 11|$$
$$+r_1r_2e^{-i\theta}|01\rangle\langle 00|+r_2^2|01\rangle\langle 01|+r_2r_3|01\rangle\langle 10|+r_2r_4|01\rangle\langle 11|$$
$$+r_1r_3e^{-i\theta}|10\rangle\langle 00|+r_2r_3|10\rangle\langle 01|+r_3^2|10\rangle\langle 10|+r_3r_4|10\rangle\langle 11|$$
$$+r_1r_4e^{-i\theta}|11\rangle\langle 00|+r_2r_4|11\rangle\langle 01|+r_3r_4|11\rangle\langle 10|+r_4^2|11\rangle\langle 11|,$$

$$\rho_A=tr_{BC}(\rho_{ABC})=r_0^2|0\rangle\langle 0|+r_0r_1e^{-i\theta}|0\rangle\langle 1|+r_0r_1e^{i\theta}|1\rangle\langle 0|+(r_1^2+r_2^2+r_3^2+r_4^2)|1\rangle\langle 1|,$$

$$\rho_B=tr_{AC}(\rho_{ABC})=(r_0^2+r_1^2+r_2^2)|0\rangle\langle 0|+(r_1r_3e^{i\theta}+r_2r_4)|0\rangle\langle 1|+(r_1r_3e^{-i\theta}+r_2r_4)|1\rangle\langle 0|+(r_3^2+r_4^2)|1\rangle\langle 1|,$$

$$\rho_C=tr_{AB}(\rho_{ABC})=(r_0^2+r_1^2+r_3^2)|0\rangle\langle 0|+(r_1r_2e^{i\theta}+r_3r_4)|0\rangle\langle 1|+(r_1r_2e^{-i\theta}+r_3r_4)|1\rangle\langle 0|+(r_2^2+r_4^2)|1\rangle\langle 1|.$$

therefore, we present the following matrix forms of all the above density operators

$$\rho_{ABC}=\begin{pmatrix} r_0^2 & 0 & 0 & 0 & r_0r_1e^{-i\theta} & r_0r_2 & r_0r_3 & r_0r_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ r_0r_1e^{i\theta} & 0 & 0 & 0 & r_1^2 & r_1r_2e^{i\theta} & r_1r_3e^{i\theta} & r_1r_4e^{i\theta} \\ r_0r_2 & 0 & 0 & 0 & r_1r_2e^{-i\theta} & r_2^2 & r_2r_3 & r_2r_4 \\ r_0r_3 & 0 & 0 & 0 & r_1r_3e^{-i\theta} & r_2r_3 & r_3^2 & r_3r_4 \\ r_0r_4 & 0 & 0 & 0 & r_1r_4e^{-i\theta} & r_2r_4 & r_3r_4 & r_4^2 \end{pmatrix},$$

$$\rho_{AB}=\begin{pmatrix} r_0^2 & 0 & r_0r_1e^{-i\theta} & r_0r_3 \\ 0 & 0 & 0 & 0 \\ r_0r_1e^{i\theta} & 0 & r_1^2+r_2^2 & r_1r_3e^{i\theta}+r_2r_4 \\ r_0r_3 & 0 & r_1r_3e^{-i\theta}+r_2r_4 & r_3^2+r_4^2 \end{pmatrix},\quad \rho_{AC}=\begin{pmatrix} r_0^2 & 0 & r_0r_1e^{-i\theta} & r_0r_2 \\ 0 & 0 & 0 & 0 \\ r_0r_1e^{i\theta} & 0 & r_1^2+r_3^2 & r_1r_2e^{i\theta}+r_3r_4 \\ r_0r_3 & 0 & r_1r_2e^{-i\theta}+r_3r_4 & r_2^2+r_4^2 \end{pmatrix},$$

$$\rho_{BC}=\begin{pmatrix} r_0^2+r_1^2 & r_1r_2e^{i\theta} & r_1r_3e^{i\theta} & r_1r_4e^{i\theta} \\ r_1r_2e^{-i\theta} & r_2^2 & r_2r_3 & r_2r_4 \\ r_1r_3e^{-i\theta} & r_2r_3 & r_3^2 & r_3r_4 \\ r_1r_4e^{-i\theta} & r_2r_4 & r_3r_4 & r_4^2 \end{pmatrix},\quad \rho_A=\begin{pmatrix} r_0^2 & r_0r_1e^{-i\theta} \\ r_0r_1e^{i\theta} & r_1^2+r_2^2+r_3^2+r_4^2 \end{pmatrix},$$

$$\rho_B=\begin{pmatrix} r_0^2+r_1^2+r_2^2 & r_1r_3e^{i\theta}+r_2r_4 \\ r_1r_3e^{-i\theta}+r_2r_4 & r_3^2+r_4^2 \end{pmatrix},\qquad \rho_C=\begin{pmatrix} r_0^2+r_1^2+r_3^2 & r_1r_2e^{i\theta}+r_3r_4 \\ r_2r_1e^{-i\theta}+r_4r_3 & r_2^2+r_4^2 \end{pmatrix}. \tag{1.2}$$

F. Liu et al. [3] discussed the inequality of super-additive relation and strong super-additive relation of three qubit pure state under l_1 norm coherence C_{l_1}. They gave the following two theorems of super-additive relation and strong super-additive relations respectively, but their proofs are not correct, so we first present the corresponding correct proofs.

Theorem 1[3] *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there is the following super-additive relation under* C_{l_1},

$$C_{l_1}(\rho_{ABC}) \geq C_{l_1}(\rho_A) + C_{l_1}(\rho_B) + C_{l_1}(\rho_C).$$

Proof. Recall that the definition of the l_1 norm of coherence[6] is as follows,

$$C_{l_1}(\rho) = \sum_{i \neq j} |\rho_{ij}|. \tag{1.3}$$

Then according to the corresponding density matrices in (1.2) , we get that

$$\begin{aligned}
C_{l_1}(\rho_A) &= 2r_0r_1, \\
C_{l_1}(\rho_B) &= 2\sqrt{r_1^2r_3^2 + r_2^2r_4^2 + 2r_1r_2r_3r_4\cos\theta}, \\
C_{l_1}(\rho_C) &= 2\sqrt{r_1^2r_2^2 + r_3^2r_4^2 + 2r_1r_2r_3r_4\cos\theta}, \\
C_{l_1}(\rho_{AB}) &= 2r_0r_1 + 2r_0r_3 + 2\sqrt{r_1^2r_3^2 + r_2^2r_4^2 + 2r_1r_2r_3r_4\cos\theta}, \\
C_{l_1}(\rho_{AC}) &= 2r_0r_1 + 2r_0r_2 + 2\sqrt{r_1^2r_2^2 + r_3^2r_4^2 + 2r_1r_2r_3r_4\cos\theta}, \\
C_{l_1}(\rho_{BC}) &= 2r_1r_2 + 2r_1r_3 + 2r_1r_4 + 2r_2r_3 + 2r_2r_4 + 2r_3r_4, \\
C_{l_1}(\rho_{ABC}) &= 2r_0r_1 + 2r_0r_2 + 2r_0r_3 + 2r_0r_4 + 2r_1r_2 + 2r_1r_3 + 2r_1r_4 + 2r_2r_3 + 2r_2r_4 + 2r_3r_4.
\end{aligned} \tag{1.4}$$

then we have

$$\begin{aligned}
&C_{l_1}(\rho_{ABC}) - C_{l_1}(\rho_A) - C_{l_1}(\rho_B) - C_{l_1}(\rho_C) \\
&= 2r_0(r_2 + r_3 + r_4) + 2r_1r_4 + 2r_2r_3 + 2(r_1r_3 + r_2r_4) \\
&\quad - 2\sqrt{r_1^2r_3^2 + r_2^2r_4^2 + 2r_1r_2r_3r_4\cos\theta} + 2(r_1r_2 + r_3r_4) - 2\sqrt{r_1^2r_2^2 + r_3^2r_4^2 + 2r_1r_2r_3r_4\cos\theta},
\end{aligned}$$

since $r_i \geq 0$ and

$$\begin{aligned}
2(r_1r_3 + r_2r_4) &\geq 2\sqrt{r_1^2r_3^2 + r_2^2r_4^2 + 2r_1r_2r_3r_4\cos\theta}, \\
2(r_1r_2 + r_3r_4) &\geq 2\sqrt{r_1^2r_2^2 + r_3^2r_4^2 + 2r_1r_2r_3r_4\cos\theta},
\end{aligned} \tag{1.5}$$

then

$$C_{l_1}(\rho_{ABC})-C_{l_1}(\rho_A)-C_{l_1}(\rho_B)-C_{l_1}(\rho_C)\geq 0,$$

so

$$C_{l_1}(\rho_{ABC})\geq C_{l_1}(\rho_A)+C_{l_1}(\rho_B)+C_{l_1}(\rho_C).$$

□

Theorem 2[3] *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there is the following strong super-additive relation under* C_{l_1},

$$C_{l_1}(\rho_{ABC})\geq C_{l_1}(\rho_A)+C_{l_1}(\rho_B)+C_{l_1}(\rho_C)+C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{AC})+C_{l_1}(\rho_{BC})$$

if $r_0r_4\geq 2r_1(r_0+r_2+r_3)+r_4(r_2+r_3)$.

Proof. According to the formulas in (1.4), we get that

$$\begin{aligned}&C_{l_1}(\rho_{ABC})-C_{l_1}(\rho_A)-C_{l_1}(\rho_B)-C_{l_1}(\rho_C)-C_{l_1}(\rho_{AB})-C_{l_1}(\rho_{AC})-C_{l_1}(\rho_{BC})\\&=2r_0r_4-4r_0r_1-4\sqrt{r_1^2r_3^2+r_2^2r_4^2+2r_1r_2r_3r_4\cos\theta}-4\sqrt{r_1^2r_2^2+r_3^2r_4^2+2r_1r_2r_3r_4\cos\theta},\end{aligned}$$

since $r_i\geq 0,\ r_0r_4\geq 2r_1(r_0+r_2+r_3)+r_4(r_2+r_3)$ and two formulas in (1.5), we have

$$C_{l_1}(\rho_{ABC})-C_{l_1}(\rho_A)-C_{l_1}(\rho_B)-C_{l_1}(\rho_C)-C_{l_1}(\rho_{AB})-C_{l_1}(\rho_{AC})-C_{l_1}(\rho_{BC})\geq 0,$$

so

$$C_{l_1}(\rho_{ABC})\geq C_{l_1}(\rho_A)+C_{l_1}(\rho_B)+C_{l_1}(\rho_C)+C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{AC})+C_{l_1}(\rho_{BC}),$$

and $C_{l_1}(\rho_{ABC})$ is strong super-additive. □

III. Sub-additive relation

In this section we mainly verify the application of the generalized Schmidt decomposition of pure state in the sub-additive relations.

P. Y. Li et al. [4] have discussed one sub-additive relation as follows,

Lemma 1[4] *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there is the following sub-additive relation under* C_{l_1},

$$C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{AC})\geq C_{l_1}(\rho_A)+C_{l_1}(\rho_B).$$

We then discuss other sub-additive relations.

Theorem 3 *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there are the following two sub-additive relations under* C_{l_1},

(i) $C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{BC})\geq C_{l_1}(\rho_A)+C_{l_1}(\rho_B)$,

(ii) $C_{l_1}(\rho_{ABC})+C_{l_1}(\rho_B)\geq C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{BC})$.

Proof. (i) According to the formulas in (1.4), we get that

$$C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{BC})-C_{l_1}(\rho_A)+C_{l_1}(\rho_B)=2r_0r_3+2r_1(r_2+r_3+r_4)+2r_2(r_3+r_4)+2r_3r_4,$$

since $r_i\geq 0$ and the two formulas in (1.5), we have

$$C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{BC})-C_{l_1}(\rho_A)-C_{l_1}(\rho_B)\geq 0,$$

hence $C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{BC})\geq C_{l_1}(\rho_A)+C_{l_1}(\rho_B)$.

(ii) According to the formulas in (1.4), we get that

$$C_{l_1}(\rho_{ABC})+C_{l_1}(\rho_B)-C_{l_1}(\rho_{AB})-C_{l_1}(\rho_{BC})=2r_0(r_2+r_4),$$

since $r_i\geq 0$ and the two formula in (1.5), we have

$$C_{l_1}(\rho_{ABC})+C_{l_1}(\rho_B)-C_{l_1}(\rho_{AB})-C_{l_1}(\rho_{BC})\geq 0,$$

hence

$$C_{l_1}(\rho_{ABC})+C_{l_1}(\rho_B)\geq C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{BC}).$$

□

IV. Monogamy relations

Using the generalized Schmidt decomposition of pure state, we discuss the monogamy relation under two coherence measures: l_1 norm coherence, C_{l_1} and the square of l_2 norm coherence, $C_{l_2}{}^2$.

Theorem 4 *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there is the following monogamy relation under* C_{l_1},

$$C_{l_1}(\rho_{ABC}) \geq C_{l_1}(\rho_{AB}) + C_{l_1}(\rho_{AC})$$

if $r_4 \geq r_1$.

Proof According to the formulas in (1.4), we get that

$$C_{l_1}(\rho_{ABC}) - C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC}) = 2r_0(r_4 - r_1) + 2r_1r_4 + 2r_2r_3 + 2(r_1r_3 + r_2r_4) - 2\sqrt{r_1^2r_3^2 + r_2^2r_4^2 + 2r_1r_2r_3r_4\cos\theta}$$
$$+ 2(r_1r_2 + r_3r_4) - 2\sqrt{r_1^2r_2^2 + r_3^2r_4^2 + 2r_1r_2r_3r_4\cos\theta},$$

since $r_i \geq 0$, $r_4 \geq r_1$ and the two formula in (1.5), we have

$$C_{l_1}(\rho_{ABC}) - C_{l_1}(\rho_{AB}) - C_{l_1}(\rho_{AC}) \geq 0,$$

hence

$$C_{l_1}(\rho_{ABC}) \geq C_{l_1}(\rho_{AB}) + C_{l_1}(\rho_{AC}).$$

Theorem 5 *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there is the following monogamy relation under* $C_{l_2}{}^2$,

$$C_{l_2}(\rho_{ABC})^2 \geq C_{l_2}(\rho_{AB})^2 + C_{l_2}(\rho_{AC})^2,$$

if $r_4^2 \geq r_1^2$.

Proof. Recall that the l_2 norm of coherence[6] is defined as

$$C_{l_2}(\rho)=\sqrt{\sum_{i\neq j}|\rho_{ij}|^2}, \tag{1.6}$$

then according to the formulas in (1.2), we can easily get that

$$\begin{aligned}
&C_{l_2}{}^2(\rho_A)=2r_0{}^2r_1{}^2,\\
&C_{l_2}{}^2(\rho_B)=2\left(r_1^2r_3^2+r_2^2r_4^2+2r_1r_2r_3r_4\cos\theta\right),\\
&C_{l_2}{}^2(\rho_C)=2\left(r_1^2r_2^2+r_3^2r_4^2+2r_1r_2r_3r_4\cos\theta\right),\\
&C_{l_2}{}^2(\rho_{AB})=2r_0{}^2\left(r_1{}^2+r_3{}^2\right)+2\left(r_1^2r_3^2+r_2^2r_4^2+2r_1r_2r_3r_4\cos\theta\right),\\
&C_{l_2}{}^2(\rho_{AC})=2r_0{}^2\left(r_1{}^2+r_2{}^2\right)+2\left(r_1^2r_2^2+r_3^2r_4^2+2r_1r_2r_3r_4\cos\theta\right),\\
&C_{l_2}{}^2(\rho_{BC})=2r_1{}^2\left(r_2{}^2+r_3{}^2+r_4{}^2\right)+2r_2{}^2\left(r_3{}^2+r_4{}^2\right)+2r_3{}^2r_4{}^2,\\
&C_{l_2}{}^2(\rho_{ABC})=2r_0{}^2\left(r_1{}^2+r_2{}^2+r_3{}^2+r_4{}^2\right)+2r_1{}^2\left(r_2{}^2+r_3{}^2+r_4{}^2\right)+2r_2{}^2\left(r_3{}^2+r_4{}^2\right)+2r_3{}^2r_4{}^2.
\end{aligned} \tag{1.7}$$

then we have

$$C_{l_2}{}^2(\rho_{ABC})-C_{l_2}{}^2(\rho_{AB})-C_{l_2}{}^2(\rho_{AC})=2r_0{}^2\left(r_4{}^2-r_1{}^2\right)+2r_1{}^2r_4{}^2+2r_2{}^2r_3{}^2,$$

since $r_i\geq 0, r_4{}^2\geq r_1{}^2$, then

$$C_{l_2}(\rho_{ABC})^2-C_{l_2}(\rho_{AB})^2-C_{l_2}(\rho_{AC})^2\geq 0,$$

hence

$$C_{l_2}(\rho_{ABC})^2\geq C_{l_2}(\rho_{AB})^2+C_{l_2}(\rho_{AC})^2. \qquad \square$$

V. Other trade-off relations

Z. M. Jiang et al. [9] studied other trade-off relations beyond monogamy relations. We now use the form of generalized Schmidt decomposition to verify those coherence inequalities from different viewpoint.

Theorem 6 *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there is the following trade-off relation under* C_{l_1},

$$C_{l_1}(\rho_{ABC}) \geq \frac{C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{AC})+C_{l_1}(\rho_{BC})}{2}.$$

Proof. According to the formulas in (1.4), we get that

$$\begin{aligned} &C_{l_1}(\rho_{ABC})-\left(\frac{C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{AC})+C_{l_1}(\rho_{BC})}{2}\right) \\ &= 2r_0r_4+r_0(r_2+r_3)+r_1r_4+r_2r_3+(r_1r_3+r_2r_4)-\sqrt{r_1^2r_3^2+r_2^2r_4^2+2r_1r_2r_3r_4\cos\theta} \\ &\quad +(r_1r_2+r_3r_4)-\sqrt{r_1^2r_2^2+r_3^2r_4^2+2r_1r_2r_3r_4\cos\theta}, \end{aligned}$$

since $r_i \geq 0$ and the formulas in (1.5), then

$$C_{l_1}(\rho_{ABC})-\left(\frac{C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{AC})+C_{l_1}(\rho_{BC})}{2}\right) \geq 0,$$

therefore,

$$C_{l_1}(\rho_{ABC}) \geq \frac{C_{l_1}(\rho_{AB})+C_{l_1}(\rho_{AC})+C_{l_1}(\rho_{BC})}{2}. \qquad \square$$

Theorem 7 *For any three-qubit pure state* $|\psi\rangle_{ABC}$ *in (1.1), there is the following trade-off relation under* $C_{l_2}{}^2$,

$$C_{l_2}{}^2(\rho_{ABC}) \geq \frac{C_{l_2}{}^2(\rho_{AB})+C_{l_2}{}^2(\rho_{AC})+C_{l_2}{}^2(\rho_{BC})}{2}.$$

Proof. According to the formulas in (1.7), we get that

$$\begin{aligned} &C_{l_2}{}^2(\rho_{ABC})-\frac{C_{l_2}{}^2(\rho_{AB})+C_{l_2}{}^2(\rho_{AC})+C_{l_2}{}^2(\rho_{BC})}{2} \\ &= r_0^2(r_2^2+r_3^2+r_4^2)+r_1^2(r_2^2+r_3^2+2r_4^2)+r_2^2(2r_3^2+r_4^2)+r_3^2r_4^2, \end{aligned}$$

since $r_i \geq 0$ we have

$$C_{l_2}{}^2(\rho_{ABC})-\frac{C_{l_2}{}^2(\rho_{AB})+C_{l_2}{}^2(\rho_{AC})+C_{l_2}{}^2(\rho_{BC})}{2} \geq 0,$$

So

$$C_{l_2}{}^2(\rho_{ABC}) \geq \frac{C_{l_2}{}^2(\rho_{AB}) + C_{l_2}{}^2(\rho_{AC}) + C_{l_2}{}^2(\rho_{BC})}{2}. \qquad \square$$

REFERENCES

[1] A. Acin, A. Andrianov, L. Costa, E. Jan, J. I. Latorre, R. Tarrach. Generalized Schmidt Decomposition and Classification of Three-Quantum-Bit States. Phys. Rev. Lett. 85, 1560(2000).

[2] H. A. Carteret, A. Higuchi, A. Sudbery. Multipartite generalization of the Schmidt decomposition. J. Math. Phys. 41, 7932 (2000).

[3] F. Liu, F. Li. Super-Additivity Relations and Decay of the l_1-Norm of Coherence. Int. J. Theor. Phys. 56: 494-502(2017).

[4] P. Y. Li, F. Liu, Y. Q. Xu. Super-Additivity Relations of the l_1-Norm of Coherence. Quant. Inf. Proc. 17,18 (2018).

[5] T. Baumgratz, M. Cramer, M. B. Plenio. Quantifying coherence. Phys. Rev. Lett. 113, 140401 (2014).

[6] Y. Y. Song, Y. H. Tao.: Notes on l_2 -Norm of Coherence. Int. J. Theor. Phys. 59: 851-860 (2020).

[7] A. Streltsov, U. Singh, H. S. Dhar, M. N. Bera, G. Ạdesso. Measuring quantum coherence with entanglement. Phys. Rev. Lett. 115, 020403 (2015).

[8] C. S. Yu.: Quantum coherence via skew information and its polygamy. Phys. Rev. A. 95, 042337(2017).

[9]Z. M. Jiang, T. G. Zhang, X. F. Huang, S. M. Fei.Trade-off relations of l_1-Norm Coherence for Multipartite Systems. Quant. Inf. Proc. 19, 92(2020) .