NON-COMMUTING OPERATORS AND SPECTRAL INVARIANCE: NON-HERMITIAN COMPLEX HAMILTONIAN AND NON-HERMITIAN REAL HAMILTONIAN

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ABSTRACT

We notice spectral invariance exists between non-commuting, non-Hermitian (PT-symmetry) Hamiltonian and corresponding real non-Hermitian (T-symmetry)operators. As an example we consider an analytically solvable model using the unified algebraic meth of (H.B. Zhang, G.Y. Jiang and G.C. Wang J. Math. Phys 56,072103 (2015)) and numerical models. Interestingly the T-symmetry operators are generated using $x \rightleftharpoons p$.

1. INTRODUCTION

Since the developments of Quantum Mechanics [1], spectral analysis plays a major role in understanding beauty behind quantum formulations. One of the fundamental quantum postulates is that" commuting operators reflect wave function invariance" mathematically

$$[H_1, H_2] = 0 \tag{1}$$

Here $H_1 \rightarrow E_1$; $H_2 \rightarrow E_2$ and $E_1 \neq E_2$. However till now no such literature which deals with spectral invariance with

$$[H_1, H_2] \neq 0 \tag{2}$$

having different nature of wave functions. However in this paper we would like to report that above relation can yield spectral invariance nature, when recasted as

$$[H_1, (x, p), H_2 = H_1 (x \rightleftharpoons p)] \neq 0 \tag{3}$$

In order to address the above relation we discuss real spectra in complex systems [2]. As such it is an abstract mathematics. However subsequent mathematical findings using Quantum Mechanics are verified experimentally both in real analysis [1] and so also in complex analysis [2]. In fact complex nature of quantum mechanics are basically due to space-time (PT) invariant nature of Hamiltonians satisfying the condition [3]

$$PTH (PT)^{-1} = H \tag{4}$$

Here the complex operator H is not necessarily be hermitian i.e.

$$H \neq H^{\dagger} \qquad (5)$$

In above P stands for parity operator or space reflection operator having the behaviour

$$PpP^{-1} = -p \qquad (6)$$

$$PxP^{-1} = -x \qquad (7)$$

Similarly T stands for time - reversal operator

$$TxT^{-1} = -x \qquad (8)$$

$$TpT^{-1} = -p \qquad (9)$$

where x (co-ordinate) and p(momentum) satisfies the commutation relation

$$[x, p] = I \qquad (10)$$

Similarly T-symmetry operator means

$$THT^{-1} = H \qquad (11)$$

Mathematically T-symmetry operator can be written as

$$[H, T] = 0 \qquad (12)$$

In this paper we present few model operators on PT-symmetry in nature and generate corresponding T symmetry operator and study the spectral nature as follows.

II. ORIGIN OF T-SYMMETRY OPERATOR

The interesting part of the above symmetry is that on replacing

$$x \rightarrow p;\ p \rightarrow x \qquad (13)$$

the magnitude of commutation relation does not change i.e

$$[p, x] = -i \rightarrow |[p, x]| = 1 \qquad (14)$$

This property has been highlighted in the case of Hermitian operators [4]. However in the case of complex operator

$$ix \rightarrow ip \qquad (15)$$

the left hand side is PT symmetry and complex in nature, however the right hand side becomes real. Physically a complex PT-symmetry operator is equivalent to real non-hermitian operator.

III. ANALYTICALLY SOLVABLE MODEL

Let us consider the model operator discussed earlier by Bender and Boettecher [2] as

$$H_1 = p^2 + x^2 + ix \tag{16}$$

having energy eigenvalues

$$E_n = 2n + \frac{5}{4} \tag{17}$$

On using $x \rightleftharpoons$, the above operator can be written as

$$h_1 = p^2 + x^2 + ip \tag{18}$$

In order to find the energy eigenvalues, we use the general relation as follows. For general quadratic operator [5]

$$H_{Ref} = h_{11}p^2 + h_{22}x^2 + ih_{12}(xp + px) + ih_1p + h_2x \tag{19}$$

having energy eigenvalue

$$\in_n = (2n+1)\sqrt{\left(h_{11}h_{22} + h_{12}^2\right)} + \frac{h_1^2h_{22} - h_2^2h_{11} - 2h_1h_2h_{12}}{4\left(h_{11}h_{22} + h_{12}^2\right)} \tag{20}$$

Following the above we have

$$h_{11} = H_{22} = 1;\ h_{12} = 0;\ h_1 = 2,\ h_2 = 0 \tag{21}$$

Under the aboce constraints

$$\in = 2n + \frac{5}{4} \tag{22}$$

It is easy to see that

$$\in = 2n + \frac{5}{4} = E_n \tag{23}$$

Hence we conclude that under the change $x \rightleftharpoons p$ the spectra remains invariant.

IV. NON-ANALYTICAL MODELS

In this case we consider few non-analytical model PT symmetry operators and corresponding T symmetry operators. In all the cases we use matrix diagonalization method to present spectral nature [4, 6] In the present, we solve the eigenvalue relation

$$H|\Phi> = E|\Phi> \quad (24)$$

where

$$|\Phi>=\sum_m A_m |m> \quad (25)$$

and |m > is the harmonic oscillator wave function, satisfying the relation

$$< m|H_0|m > = < m[p^2 + x^2]|m > = (2m + 1) \quad (26)$$

A-1 : Broken spectra

The Hamiltonian considered are the following

$$H_2 = p^2 + |x| + ix \quad (27)$$

$$h_2 = x^2 + |p| + ip \quad (28)$$

A-2 : Unbroken spectra

The Hamiltonian considered are the following

$$H_3 = p^2 + x^4 + ix \quad (29)$$

$$h_3 = x^2 + p^4 + ip \quad (30)$$

A-3: Broken spectra

The Hamiltonian considered are the following

$$H_4 = p^2 + x^4 + i10x \quad (31)$$

$$h_4 = x^2 + p^4 + i10p \quad (32)$$

A-5: Broken spectra

The Hamiltonian considered are the following

$$H_5 = p^2 + ix\,|x| \quad (33)$$

$$h_5 = x^2 + ip\,|p| \quad (34)$$

Few eigenvalues are tabulated in Table-1

V. CONCLUSION

In this paper a reader will notice that all PT-symmetry operator is associated with T-symmetry operator reflecting iso-spectral nature. Similarly our previous study on Hermitian operator also reflects the same thing [5] i.e. every Hermitian operator is associated with another Hermitian operator having the same spectra. In conclusion non-commuting operators generated using $x \rightleftharpoons p$ will always yield same spectral nature (see figs 1-4). As stated above the above Hamiltonians correspond to different nature of wave functions with a view to justify this nature, we consider the unbroken spectra and display the wave function mod square in fig-5 and 6.

REFERENCES

1. B.H. Bransden and C.J. Joachain, Quantum Mechanics, Pearson, Dorling Kindersley (India) Pvt.Ltd (2000).

2. C.M. Bender and S. Boettecher, Real spectra in non-Hermitian Hamiltonian having PT-symmetry, Phys.Rev.Lett 80, 5243(1998).

3. C.M. Bender, D.W. Hook and L.R. Mead,Conjucture on the analyticity of PT-symmetric potentials and the reality of their spectra, J.Phys. A 41,392005 (2008).

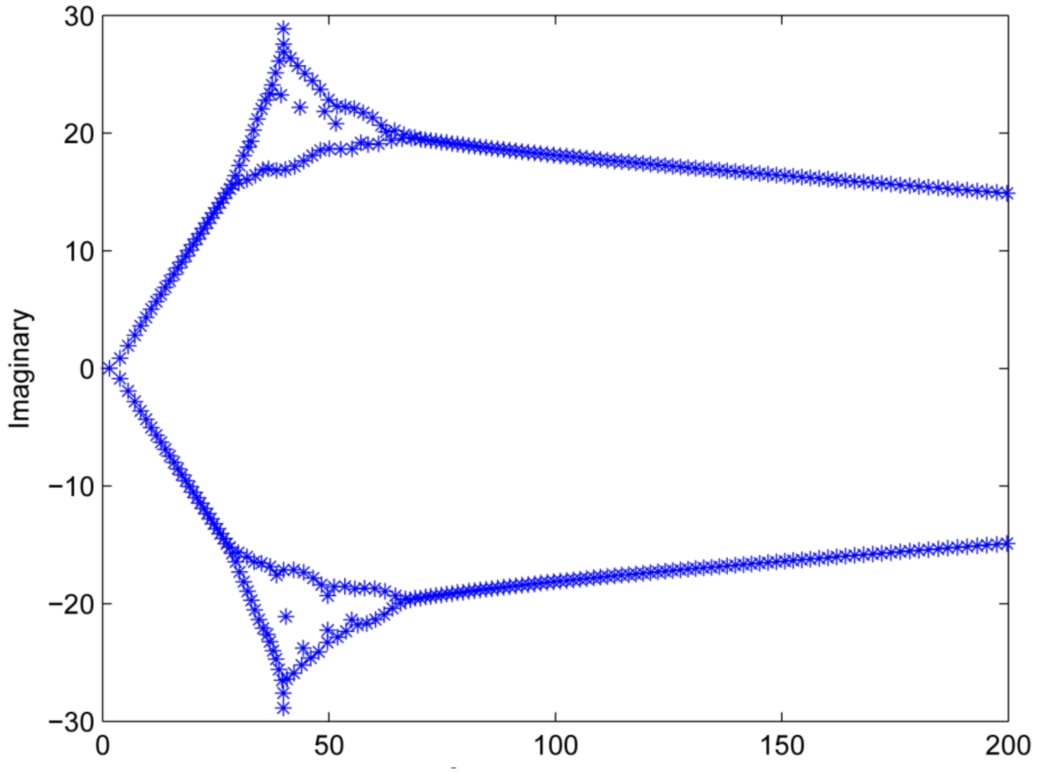
4. B.Rath Can the spectra of Hermitian operator be invariant under $x \rightleftharpoons p$: Case study;1D general oscillator, Jour. of Advances. Phys, 14(1), 5326(2018).

5. H.B. Zhang, G.Y. Jiang and G.C. Wang,Unified algebraic method to non-Hermitian systems with Lie algebraic linear structure, J.Math.Phys.56, 072103(2015).

6. L.N. Trefethen, Computation of Pseudospectra, Acta. Numerica (1999) (Cambridge University Press).

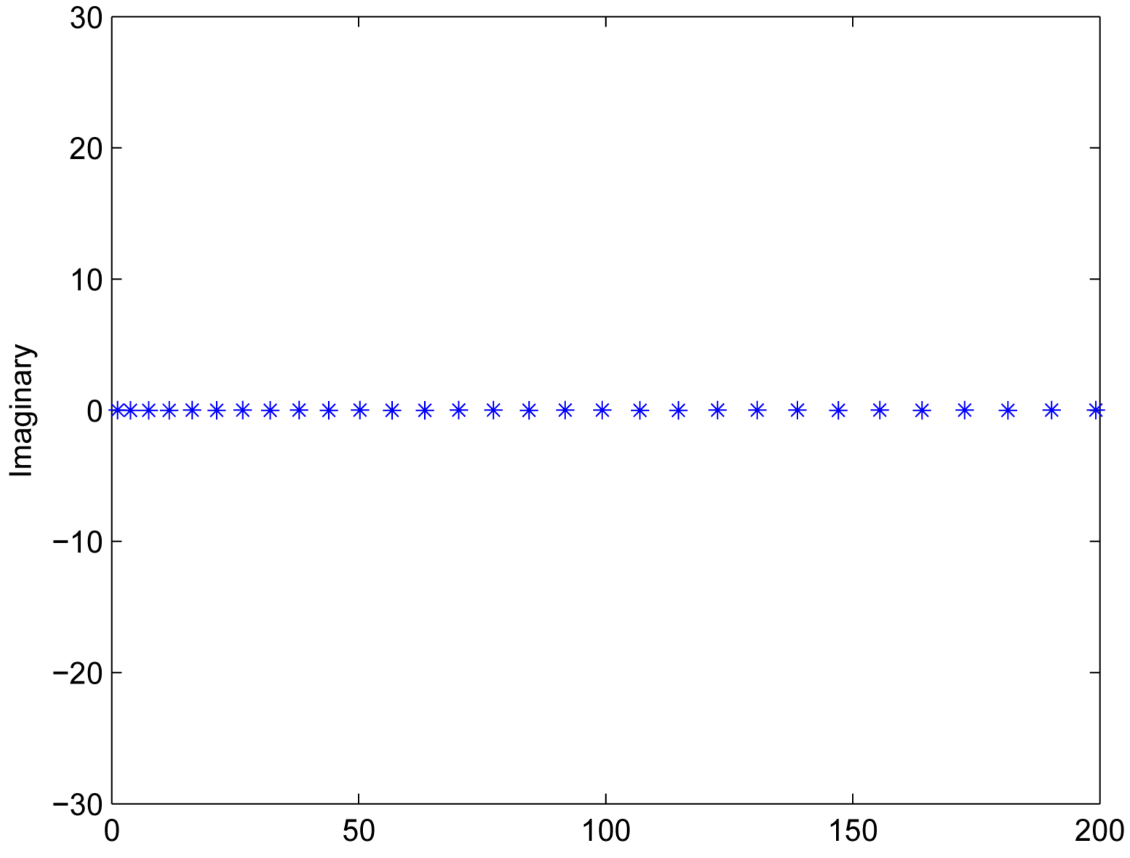
Table-1: PT-symmetry and T-symmetry operator

n	**H_1**	**h_1**	**See Fig**
0	1.25	1.25	
1	3.25	3.25	
2	5.25	5.25	
3	7.27	7.25	
4	9.25	9.25	
n	**H_2**	**h_2**	**See Fig**
0	1.488158	1.488158	Fig-1
1	3.852053 + 0.868850 i	3.852053 + 0.868850 i	
2	3.852053 – 0.868850 i	3.852053 – 0.868850 i	
3	5.637605 + 1.914109 i	5.637605 + 1.914109 i	
4	5.637605 – 1.914109 i	5.637605 – 1.914109 i	
n	**H_3**	**h_3**	**See Fig**
0	1.194489	1.194489	Fig-2
1	3.813357	3.813357	
2	7.476329	7.476329	
3	11.661074	11.661074	
4	16.275800	16.275800	
n	**H_4**	**h_4**	**See Fig**
0	7.919338 – 7.067323 i	7.919338 – 7.067323 i	Fig-3
1	7.919338 + 7.067323 i	7.919338 + 7.067323 i	
2	13.386047 – 3.187097 i	13.386047 – 3.181097 i	
3	13.386047 + 3.181097 i	13.386047 + 3.181097 i	
4	17.957084	17.957084	
n	**H_5**	**h_5**	**See Fig**
0	1.285090	1.258090	Fig-4
1	4.991283 – 0.780527 i	4.991283 – 0.780527 i	
2	4.991283 + 0.780527 i	4.991283 + 0.780527 i	
3	8.618015 + 3.363276 i	8.618015 + 3.363276 i	
4	13.386047 + 3.181097 i	13.386047 – 3.181097 i	
5	17.957084	17.957084	



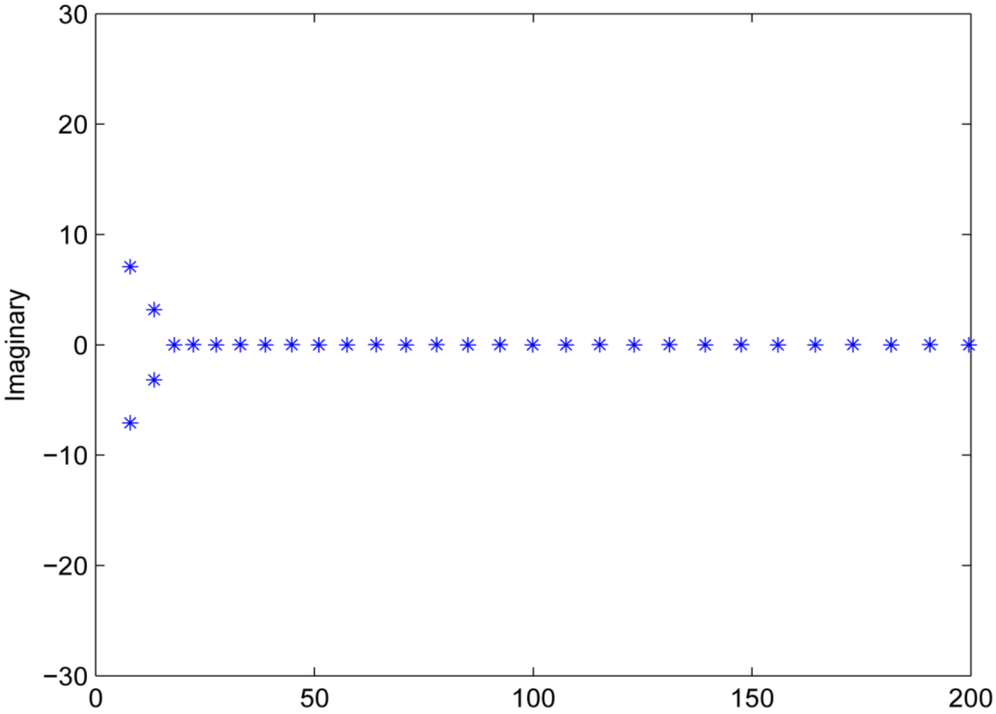
Real, $H_2 = p^2 + |x| + ix$ and $h_2 = x^2 + |p| + ip$

Figure 1 : Energy eigenvalues



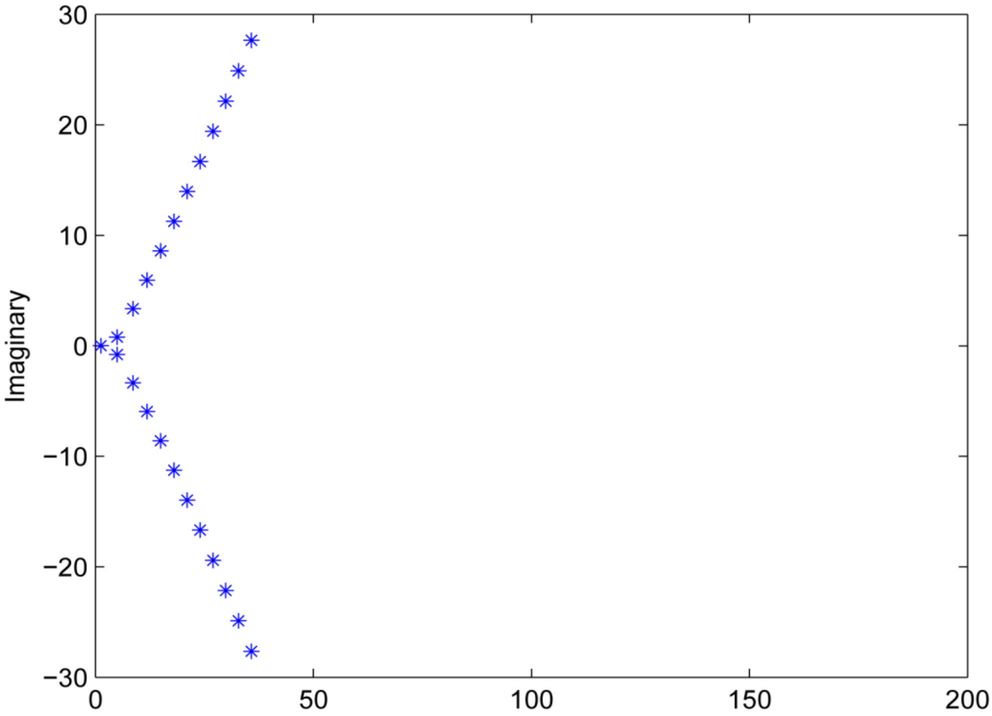
Real, $H_3 = p^2 + x^4 + ix$ and $h_3 = x^2 + p^4 + ip$

Figure 2 : Energy eigenvalues



Real, $H_4 = p^2 + x^4 + 10\,ix$ and $h_4 = x^2 + p^4 + 10\,ip$

Figure 3 : Energy eigenvalues



Real, $H_5 = p^2 + |x|\,ix$ and $h_5 = x^2 + |p|\,ip$

Figure 4 : Energy eigenvalues

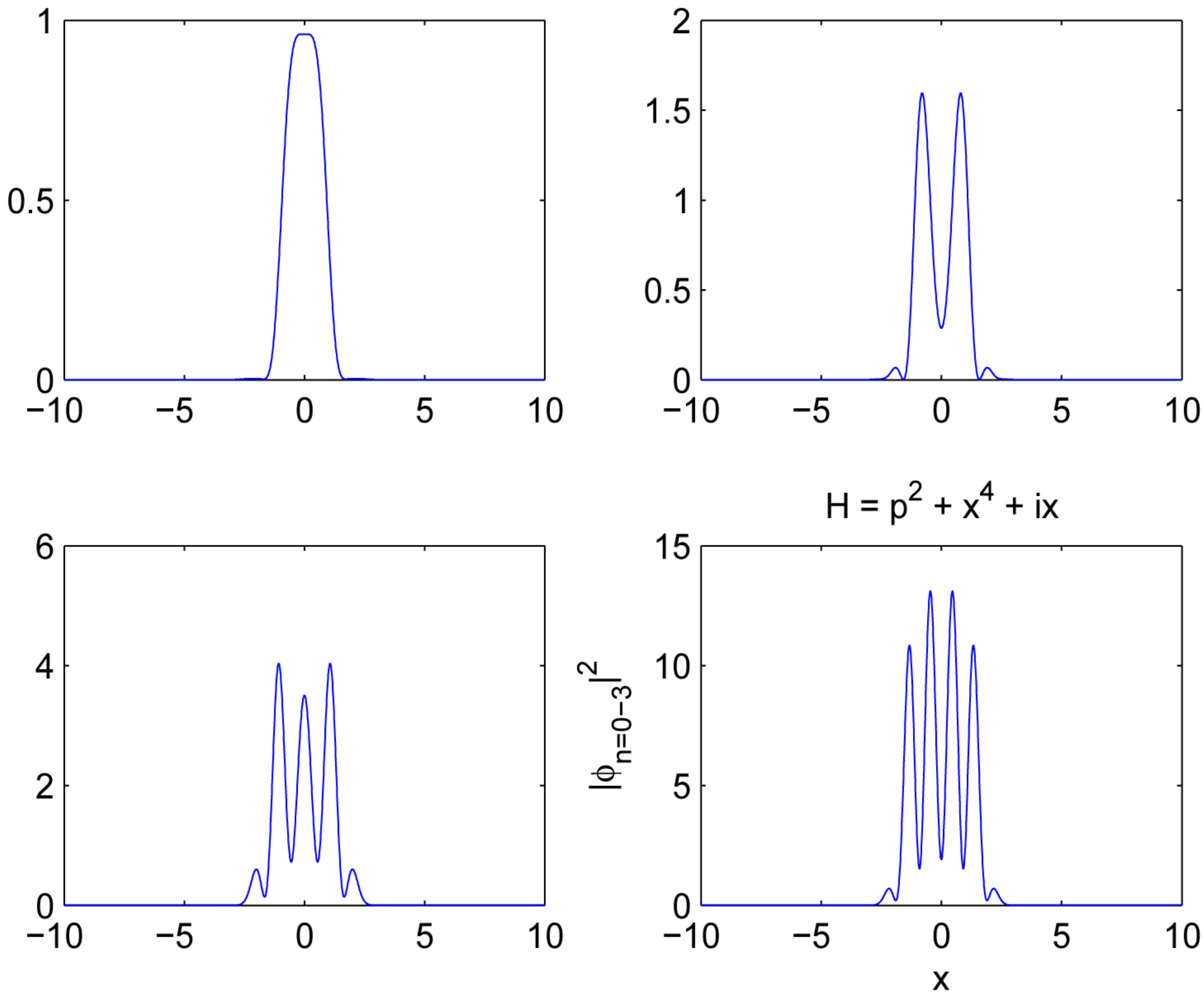


Figure 5 : Wave function mod square

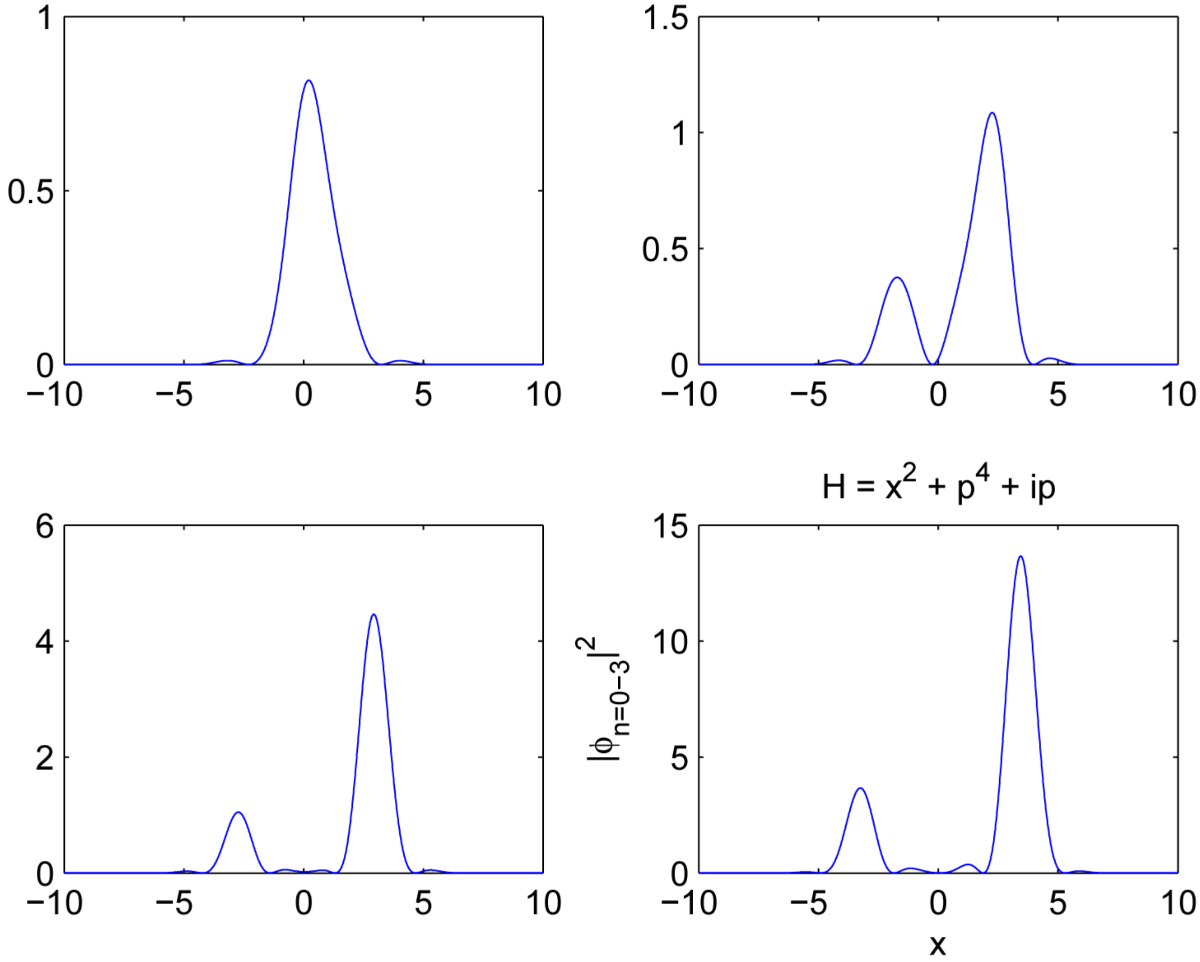


Figure 6 : Wave function mod square