ISSN 2059-9951

GENERATING (2 × 2) PT-SYMMETRIC MATRICES USING PAULI MATRICES AS PARITY OPERATOR: BROKEN SPECTRA AND STOP LIGHT IN UNBROKEN SPECTRA AT UNEQUAL POINTS

Biswanath Rath

Department of Physics, North Orissa University, Takatpur, Baripada -757003

Odisha, INDIA; E.mail: biswanathrath10@gmail.com.

ABSTRACT

We find that only real Pauli matrices can be considered as Parity operators under the direct generation of unbroken u PT-symmetric Hamiltonian. However complex form of Pauli matrix as parity operator can lead to broken PT-symmetry. We feel appropriate to correct the incorrect parity analysis reported elsewhere by some authors. Interestingly we show that present unbroken spectra consisting of unequal real eigenvalues can stop the light.

PACS no-11.30.Er

Keywords: Pauli matrices, parity operator, PT-symmetry, broken spectra, broken spectra.

1.Introduction

Pauli matrices [1]

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (1)$$

$$\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad (2)$$

and $$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (3)$$

are as old as quantum mechanics . All the Pauli matrices have eigenvalues $\lambda_{\sigma_{1,2,3}} = \pm 1$.

Probably this has motivated many authors [2-9] to discuss complex quantum mechanics(PT symmetric quantum mechanics) using P-parity operator as Pauli matrix under broken or un-broken spectra. If the spectra are broken, the model analysis becomes uninteresting. However here we just discuss how to accept or reject a Pauli matrix in discussing PT-symmetry Hamiltonian. In PT-symmetry quantum mechanics, Hamiltonian operator must satisfy the relation[2-9]

$$[H, PT] = 0 \Leftrightarrow T^{-1}(P^{-1}HP)T = H \quad (4)$$

This can be rewritten as

$$HP = PH^{*} \quad (5)$$

In this context we would like to point out that if one considers the PT-symmetry condition as

$$[PT, H] = 0 \Leftrightarrow HP = PH^{*} \quad (6)$$

then the corresponding relation between P and H remains the same as above . Further using the property of parity operator

$$P^{2} = 1 \quad (7)$$

It is easy to see that an alternate form of the above relation can be written as [6]

$$PH = H^{*}P \quad (8)$$

2. Unbroken PT-symmetry

Here we generate PT-symmetric Hamiltonians using Pauli matrices as Parity operator. The procedure is as follows. Let Hamiltonian $H_i$of an operator is PT-symmetric with parity operator as P_i then

$$[H_k, P_kT] = 0 \Leftrightarrow T^{-1}\,(P_k^{-1}H_kP_k)T = H_k \quad (9)$$

Considering a general form of H_k as

$$H_k = \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ c_1 + ic_2 & d_1 + id_2 \end{bmatrix} \quad (10)$$

where $a_{1,2}$, $b_{1,2}$and$c_{1,2}$ are unknown parameters an are determines using the relation as

$$H_kP_k = P_kH_k^{*} \quad (11)$$

Case-1 : $P_1 = \sigma_1$

Let us consider that $\sigma_1 = P_1$as the parity operator, where

$$P_1 = \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (12)$$

(a) Bender, Berry and Mandilara model[2]

In this case we find

$$H_1 = \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ b_1 - ib_2 & a_1 - ia_2 \end{bmatrix} \tag{13}$$

which is nothing but Bender, Berry and Mandilara [2] model. The eigenvalues of the above Hamiltonian H_1

$$E_{\pm}^{(1)} = a_1 \pm \sqrt{b_1^2 + b_2^2 - a_2^2} \tag{14}$$

This can be reacted as [3]

$$E_{\pm}^{(1)} = a_{Re} \pm \sqrt{|b|^2 - a_{1m}^2} \tag{15}$$

provided $b_2^1 + b_2 \geq a_2$. In fact authors have not mentioned the explicit form of parity. From above analysis one will notice $P_1 = \sigma_1$ is the only appropriate parity [2]. Now we consider different forms of Hamiltonians considering different values of $a_{1,2}$, $b_{1,2}$ as follows.

(b) Bender, Brody and Jones model[3]($b_2 = 0$)

$$H_{BBJ} = \begin{bmatrix} a_1 + ia_2 & b_1 \\ b_1 & a_1 - ia_2 \end{bmatrix} \tag{16}$$

$$E_{\pm}^{BBJ} = a_1 \pm \sqrt{b_1^2 - a_2^2} \tag{17}$$

On changing the parameters $a_1 = r\cos\theta;\ a_2 = r\sin\theta;\ b_1 = s,$ we write the above Hamiltonian as

$$H_{BBJ} = \begin{bmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{bmatrix} \tag{18}$$

having eigenvalues

$$E_{\pm}^{BBJ} = r\cos\theta \pm \sqrt{s^2 - r^2\sin^2\theta} \tag{19}$$

(c) New model $b_2 = a_2$

$$H_{new}^{c} = \begin{bmatrix} a_1 + ia_2 & b_1 + ia_2 \\ b_1 - ia_2 & a_1 - ia_2 \end{bmatrix} \tag{20}$$

having eigenvalues

$$E_{\pm}^{b} = a_1 \pm b_1 \tag{21}$$

(d) New model $b_2 = a_2$; $b_1 = 0$

$$H_{new}^{d} = \begin{bmatrix} a_1 + ia_2 & ia_2 \\ -ia_2 & a_1 - ia_2 \end{bmatrix} \tag{22}$$

having degenerate eigenvalues

$$E_{\pm}^{c} = a_1 \tag{23}$$

(e) New model $b_2 \neq a_2$; $a_1 = 0$

$$H_{new}^{e} = \begin{bmatrix} ia_2 & b_1 + ib_2 \\ b_1 - ia_2 & -ia_2 \end{bmatrix} \tag{24}$$

having degenerate eigenvalues

$$E_{\pm}^{d} = \pm b_1 \tag{25}$$

Parity analysis on Bagchi and Barik model [9]

In this analysis, we notice Bagchi and Barik [8] have used the PT-symmetry operator

$$H_{BB} = \begin{bmatrix} i\gamma & -\zeta \\ -\zeta & -i\zeta \end{bmatrix} \tag{26}$$

The corresponding parity operator using the present analysis is found to be

$$P_{BB} = -\sigma_1 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{27}$$

The above correct form can be realised using C-symmetry and parity analysis by Wang [7]. However Bagchi and Barik [9] claim that parity operator should beσ_1.Let us pay our attention on other form of parity operator, that yield real spectra considering a different Pauli matrix as follows.

Case-2: $P_3 = \sigma_3$

Now consider the other real operator $\sigma_3$as Parity operator.

$$P_3 = \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{28}$$

In this case, considering the nature of P_3 and symmetry, the appropriate Hamiltonian becomes

$$H_3 = \begin{bmatrix} a_1 & ib_2 \\ ib_2 & -a_1 \end{bmatrix} \tag{29}$$

The corresponding eigenvalues are

$$E_{\pm}^{(3)} = \pm\sqrt{a_1^2 - b_2^2} \tag{30}$$

provided $a_1 \geq b_2$. In this case if one consider, a typicase as $a_1 = b_2$,the operator is still PT-symmetric in nature i.e.

$$H^{(3)} = \begin{bmatrix} a_1 & ia_1 \\ ia_1 & -a_1 \end{bmatrix} \tag{31}$$

but having zero eigenvalues. Mathematically it sounds good and hope it can be realised. Under this nature of parity operator, we discuss the model operator proposed earlier by Wang[6] as follows.

Wang[6] model PT-symmetry Hamiltonian and corresponding correct parity analysis

In a paper Wang[6] proposed a model PT-symmetry operator as

$$H_W = \begin{bmatrix} \in + \gamma\cos\delta & -i(\gamma\sin\delta - \rho) \\ i(\gamma\sin\delta + \rho) & \in - \gamma\cos\delta \end{bmatrix} \tag{32}$$

and argued that parity operator should be related to σ_2 . In the opinion of present author, the appropriate parity operator is $P_3 = \sigma_3$ as it satisfies the relation

$$[H_w, P_3T] = 0 \tag{33}$$

i.e $$H_W P_3 = P_3 H_W^*$$ (34)

Ahmed[4] model PT-symmetry Hamiltonian and corresponding correct parity analysis[5]

In this case we discuss Ahmed[4] model of PT -operator as

$$H_{Ahmed} = \begin{bmatrix} a-c & ib \\ ib & a+c \end{bmatrix} \tag{35}$$

In this model present author notice that the correct parity operator is

$$P_{A-C} = -\sigma_3 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{36}$$

Further it is easy to notice that

$$H_{Ahmed} P_{A-C} = P_{A-C} H^*_{Ahmed} \tag{37}$$

Interested reader can derive the present correct form of parity operator using the C-symmetry and parity analysis [7]. At this point we would like to state that one has to use the correct form of wave function [5].

3. Broken PT-symmetry

Now consider the other complex Hermitian operator σ_2 as Parity operator. Above analysis will yield the Hamiltonian

$$H_2 = \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ -b_1 + ib_2 & a_1 - ia_2 \end{bmatrix} \tag{38}$$

The corresponding eigenvalues are

$$E^{(1)}_{\pm} = a_1 \pm i\sqrt{a_2^2 + b_1^2 + b_2^2} \tag{39}$$

In this case spectra becomes broken .In view of broken spectra, PT-symmetry is not interesting as the eigenvalues are complex. Below we will discuss few lines regarding unbroken spectra under modified form of σ_2.

Modified σ_2 under broken spectra.

Let us consider the modified form of Pauli matrix σ_3 as

Case-(i)

$$P_1^{\text{modified}} = \begin{bmatrix} \pm\cos\theta & \mp i\sin\theta \\ \pm i\sin\theta & \mp\cos\theta \end{bmatrix} \tag{40}$$

or

$$P_2^{\text{modified}} = \begin{bmatrix} \pm\sin\theta & \mp i\cos\theta \\ \pm i\cos\theta & \mp\sin\theta \end{bmatrix} \tag{41}$$

Even though eigenvalues of this modified operator is

$$P_1^{\text{modified}} = \pm 1 \tag{42}$$

still it is not suitable for discussion under un-broken spectra. In this context wepointout that the model P-parity operator by Wang [7]

$$P_+^{\text{in-wang}} = \begin{bmatrix} \cos\theta & -i\sin\theta \\ i\sin\theta & -\cos\theta \end{bmatrix}_{\theta=\pi/2} = \sigma_2 \tag{43}$$

can hardly generate any unbroken PT-symmetry operator, which will yield un-broken spectra. Further interested reader will notice that $P_+^{\text{in-wang}}$ fails to commute with

$$H_w = \begin{bmatrix} \in + \gamma\cos\theta & -i(\gamma\sin\theta - \rho) \\ i(\gamma\sin\theta + \rho) & \in - \gamma\cos\theta \end{bmatrix}_{\theta=\pi/2} \tag{44}$$

Mathematically

$$\left[H_w, P_+^{\text{modified}}\right] \neq 0 \tag{45}$$

In other words $P_+^{\text{in-wang}}$ is not the parity operator.[6]

4. Unbroken spectra to stop the light at unequal points

In this analysis we discuss a possible application of unbroken spectra to stoplight at points having unequal eigenvalues following the work of Goldzak et.al[10].Previously it has been reported that light can be stopped at exceptional points having same energy using unbroken PT-symmetry operator in the form of a (2×2) matrix. Authors have shown that the group velocity of light vanishes according to authors, the group velocity of propagation[1]

$$\nu_g = \frac{d\omega}{d\beta} = \frac{2c^2\beta\psi^2 dx}{\int\left[\partial\left(n^2\omega^2\right)/(\partial\omega\right]\psi^2 dx} \quad (46)$$

In complex space brackett relation is simplified as [1]

$$\left\langle \psi^* \mid \psi \right\rangle = \int \psi^2 dx \quad (47)$$

According to authors exceptional points will have same energy. If the constant involved in the analysis is made zero, then specific points will have zero energy. However we use our above equation

$$H_\beta = \begin{bmatrix} a_1 + ia_2 & b_1 + ib_2 \\ b_1 - ib_2 & a_1 - ia_2 \end{bmatrix}_{a_i = b_i = \beta} \quad (48)$$

Becomes

$$H_\beta = \begin{bmatrix} \beta(1+i) & \beta(1+i) \\ \beta(1-i) & \beta(1-i) \end{bmatrix} \quad (49)$$

The corresponding eigenvalues becomes unequal

$$E_{\mp} = 2\beta; 0 \quad (50)$$

The wave functions are

$$|\psi_+ >= \frac{1}{2}\begin{bmatrix} (1+i) \\ (1-i) \end{bmatrix} \quad (51)$$

and

$$\psi_- >= \frac{1}{2}\begin{bmatrix} (1-i) \\ (1+i) \end{bmatrix} \quad (52)$$

Further

$$\langle \Psi_+ | H_\beta | \Psi_+ \rangle = 0 \quad (53)$$

and

$$\left\langle \psi_- \middle| H_\beta \middle| \psi_- \right\rangle = 2\beta \quad (54)$$

ISSN 2059-9951

However in both the cases

$$\nu_g \propto \int \psi_{\pm}^2 dx = 0 \tag{55}$$

As the points are posses unequal eigenvalues, we say as “unequal points”.

5. Conclusion

We notice that Pauli matrix $\sigma_2$can not be termed as Parity operator under unbroken spectra. However$\sigma_{1,3}$ can be interpreted as parity operator under unbroken spectra. Further we also notice that the above matrix analysis can generate interest among many to verify it experimentally to stop light.

References

1. R.Shankar, Principles of Quantum Mechanics, Plenum Press(Chapter-11,section11.5)(2005).
2. C.M.Bender, M.V.Berry and A.Mandilara, Generalized PT-symmetry and real spectra. J.Phys. A.(2002),35:L467-L471.
3. C.M. Bender, D.C.Brody and H.F.Jones. Complex extension of Quantum Mechanics. Phys. Rev.Lett.(2002),89(27):270401.
4. Z.Ahmed.C,PT,CPT invariance of pseudo-Hermitian Hamiltonians.arXiv.quantph/0302141v1.
5. B.Rath,Comment on C,PT,CPT invariance of pseudo-Hermitian Hamiltonians[Z.Ahmedarxiv.quant-ph/0302141v1], arXiv:1907.05235[quant-ph].
6. Q.Wang,2×2 PT-symmetric matrices and their applications. Phil.Trans. R.Soc(2012),A371:20120045.
7. Q.Wang,S.Chia and J.Zhang. PT-Symmetry as a Generalization of Hermiticity,J.Phys.A(2010),(43):295301 .
8. B.Rath,C,P,T analysis on 1-parameter model (2x2) matrix in complex quantum systems, Eur.J.Math and Comp.Sc(2019),6(1):63-67.
9. B.Bagchi and S.Barik,Preservation of no-signalling principle in parity time symmetric quantum systems,arxiv:1904.08381 v1 [quant-ph].
10. T.Goldzak,A.A. Mailybaev and N.Moiseyev. Light stops at exceptional points,arXiv:1709.10172 v2[physics.optics](2018),Phys.Rev¿Lett.120,013901(2018).