Notes on Schatten-2 Norm of Coherence

Yiyang Song, Yuanhong Tao[1,2]

Department of Mathematics, Collage of Science, Yanbian University, Yanji, Jilin 133002

Abstract: The Schatten-2 norm of coherence is studied in this paper. Firstly, the Schatten-2 norms of coherence for some states are calculated, and their analytical formula are showed to be equal to the l_2 norm of coherence for these states. Then the Schatten-2 norms of coherence for maximally coherent states is discussed. Finally, the Schatten-2 norm of coherence is proved to be an illegitimate coherence measure using YZX framework, which is simpler than the precious method in [Phys. Rev. A. 93: 012110 (2016)].

Keywords: Schatten-2 norm of Coherence; Qubit states; Maximally coherent state

1 Introduction

Quantum coherence is an important subject in quantum theory and quantum information science. Baumgratz et al. [1] first proposed a framework (BCP framework) for quantifying quantum coherence. A function C defined on a space of quantum states can be used as a measure of coherence if the following four conditions are satisfied [1],

(B1) $C(\rho)\geq 0$, and $C(\rho)=0$ if and only if $\rho\in I$, where I is the set of incoherent states that are diagonal in a fixed basis $\{|i\rangle\}$;

(B2) $C(\Lambda(\rho))\leq C(\rho)$ for any incoherent operation Λ, i.e., Λ is completely positive trace preserving (CPTP) map, $\Lambda(\rho)=\sum_n K_n I K_n^{\dagger}\subseteq I$ for all *n*, *I* is the set of incoherent states;

(B3) $\sum_n p_n C(\rho_n)\leq C(\rho)$, where $p_n=Tr\left(K_n\rho K_n^{\dagger}\right)$, $\rho_n=K_n\rho K_n^{\dagger}/p_n$, $\{K_n\}$ is a set of incoherent Kraus operators;

(B4) C is a convex function, i.e., $C\left(\sum_i p_i\rho_i\right)\leq\sum_i p_i C(\rho_i)$ for any set of quantum states $\{\rho_i\}$ and any probability distribution $\{p_i\}$.

Many legitimate coherence measures satisfying the BCP framework were proposed in recent literature, such as l_1 -norm coherence [1], relative entropy of

[1] Corresponding author: Yuan-hong Tao, E-mail: taoyuanhong12@126.com

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coherence [1], modified trace norm of coherence [2], geometric measure of coherence [3], robustness coherence [4], skew information of coherence [5] etc. There are also other illegitimate coherence measures violating (B3) of BCP framework, such as the l_2 norm of coherence [1] and the Schatten-p norm of coherence [6].

The l_2 norm of coherence, $C_{l_2}(\rho)$, is defined as $C_{l_2}(\rho) = \min_{\delta \in I} \|\rho - \delta\|_{l_2}$, where the l_2 norm of a matrix $X = (x_{ij})$ is $\|X\|_{l_2} == \left(\sum_{ij} |x_{ij}|^2\right)^{1/2}$. And the Schatten-p norm of coherence is defined as $C_p(\rho) = \min_{\delta \in I} \|\rho - \delta\|_p$.

In the sequel we focus on the following Schatten-2 norm of coherence,

$$C_2(\rho) = \min_{\delta \in I} \|\rho - \delta\|_2,$$

here the Schatten-2 norm of a matrix $X = (x_{ij})$ is defined as follows [6],

$$\|X\|_2 = \left(Tr|X|^2\right)^{1/2} = \left(\sum_i^r \sigma_i^2\right)^{1/2},$$

where r *is the* rank of X and σ_i denotes non-zero singular values of X, i.e., eigenvalues of $|X| = \sqrt{X^\dagger X}$.

Recently, Yu et al. [2] put forward a YZX framework to quantify quantum coherence. A nonnegative function C defined on a space of quantum states can be used as a measure of coherence if the following three conditions is satisfied [2],

(C1) $C(\rho) \geq 0$, and $C(\rho) = 0$ if and only if $\rho \in I$;

(C2) $C(\Lambda(\rho)) \leq C(\rho)$ for any incoherent operation Λ;

(C3) $C(p_1\rho_1 \oplus p_2\rho_2) = p_1 C(\rho_1) + p_2 C(\rho_2)$ for block diagonal states ρ in the incoherent basis.

It has been shown that the YZX framework is equivalent to the BCP framework [2].

In this paper, we focus on the properties of Schatten-2 norm of coherence. We first calculate the Schatten-2 norms of coherence for some states, then we prove that these Schatten-2 norms of coherence are equal to their corresponding l_2 norms of coherence. Moreover, we calculate the Schatten-2 norms of coherence for maximally coherent states. Finally, we use YZX framework to prove that the Schatten-2 norm of coherence is not a legitimate coherence measure.

2 The Schatten-2 Norms of Coherence of Some States

We begin with the calculation of the Schatten-2 norms of coherence for the following three kinds of quantum states:

qubit states: $\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}$, three qutrit states: $\rho_X = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13}^* & 0 & a_{33} \end{pmatrix}$, $\rho_Y = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12}^* & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$,

$\rho_Z = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23}^* & a_{33} \end{pmatrix}$,

a class of qudit states, $p^* = \begin{pmatrix} x_1 & a & \cdots & a \\ a & x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_d \end{pmatrix}$.

we also look for their optimal incoherent states under Schatten-2 norms of coherence.

Proposition 1. For the above three kinds of quantum states, their Schatten-2 norms of coherence are equal to their l_2 norms of coherence, i.e.,

$$C_2(\rho) = C_{l2}(\rho)$$

and their optimal incoherent states under Schatten-2 norm of coherence are their diagonal states.

Proof. Recall that we have calculated the l_2 norms of coherence of all the above three kinds of qubit states, so we only need to calculate their Schatten-2 norms of coherence, and make some comparison.

Firstly, we consider the qubit states $\rho = \begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}$. Denote the qubit incoherent state as $\delta = diag\{x, 1-x\} \in I$, then

$$\rho - \delta = \begin{pmatrix} a-x & b \\ b^* & x-a \end{pmatrix}$$

and

$$(\rho-\delta)^\dagger(\rho-\delta) = \begin{pmatrix} (a-x)^* & b \\ b^* & (x-a)^* \end{pmatrix}\begin{pmatrix} a-x & b \\ b^* & x-a \end{pmatrix} = \begin{pmatrix} |a-x|^2+|b|^2 & (a-x)^* b + (x-a) b \\ (a-x) b^* + (x-a)^* b^* & |x-a|^2+|b|^2 \end{pmatrix},$$

so

$$\|\rho-\delta\|_2 = Tr((\rho-\delta)^\dagger(\rho-\delta))^{1/2} = \left(|a-x|^2 + |x-a|^2 + 2|b|^2\right)^{1/2}.$$

It is easy to see that $\|\rho-\delta\|_2 \geq \sqrt{2}|b|$, and the equality holds when $x=a$, so

$$C_2(\rho) = \min_{\delta\in I}\|\rho-\delta\|_2 = C_{l_2}(\rho) = \sqrt{2}|b|,$$

and the minimum attains at $\delta = diag\{a,1-a\} = \rho_{diag}$, i.e. the optimal incoherent state of ρ is ρ_{diag}.

Secondly, we consider the three qutrit states $\rho_X = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13}^* & 0 & a_{33} \end{pmatrix}$, $\rho_Y = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12}^* & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$, $\rho_Z = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23}^* & a_{33} \end{pmatrix}$. Denote the qutrit incoherent state as $\delta = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$, then

$$\rho_X - \delta = \begin{pmatrix} a_{11}-x & 0 & a_{13} \\ 0 & a_{22}-y & 0 \\ a_{13}^* & 0 & a_{33}-z \end{pmatrix},$$

and

$$(\rho_X-\delta)^\dagger(\rho_X-\delta) = \begin{pmatrix} |a_{11}-x|^2+|a_{13}|^2 & 0 & (a_{11}-x)^* a_{13} + (a_{33}-z)a_{13} \\ 0 & |a_{22}-y|^2 & 0 \\ (a_{11}-x)a_{13}^* + (a_{33}-z)^* a_{13}^* & 0 & |a_{33}-z|^2+|a_{13}|^2 \end{pmatrix},$$

so

$$\|\rho_X-\delta\|_2 = Tr\left((\rho_X-\delta)^\dagger(\rho_X-\delta)\right)^{1/2} = \left(|a_{11}-x|^2+|a_{22}-y|^2+|a_{33}-z|^2+2|a_{13}|^2\right)^{1/2}.$$

Obviously, $\|\rho_X-\delta\|_2 \ge \sqrt{2}|a_{13}|$, and the equality holds if and only if $x=a_{11}$, $y=a_{22}$, $z=a_{33}$,

thus

$$C_2(\rho_X) = \min_{\delta\in I}\|\rho-\delta\|_2 = C_{l_2}(\rho_X) = \sqrt{2}|a_{13}|$$

and the minimum attains at $\delta = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} = \rho_{diag}$.

Similarly, one can prove the equalities of ρ_Y and ρ_Z as follows,

$$C_2(\rho_Y) = C_{l_2}(\rho_Y) = \sqrt{2}|a_{12}|,\ C_2(\rho_Z) = C_{l_2}(\rho_Z) = \sqrt{2}|a_{23}|,$$

and their optimal incoherent states are their diagonal states ρ_{diag}^Y, ρ_{diag}^Z, respectively.

Finally, we consider the class of qudit states p* = $\begin{pmatrix} x_1 & a & \cdots & a \\ a & x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_d \end{pmatrix}$. Denote the qudit incoherent state as $\delta = \begin{pmatrix} y_1 & 0 & \cdots & 0 \\ 0 & y_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_1 \end{pmatrix}$, then

$$\rho^* - \delta = \begin{pmatrix} x_1 - y_1 & a & \cdots & a \\ a & x_2 - y_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_d - y_d \end{pmatrix}.$$

and

$$\left\|\rho^* - \delta\right\|_2 = Tr\left(\left(\rho^* - \delta\right)^\dagger \left(\rho^* - \delta\right)\right) = \left(\left|x_1 - y_1\right|^2 + \left|x_2 - y_2\right|^2 + \cdots + \left|x_d - y_d\right|^2 + \sqrt{d(d-1)}\left|a\right|^2\right)^{1/2}.$$

Obviously, $\left\|\rho^* - \delta\right\|_2 \ge \sqrt{d(d-1)}\left|a\right|$, and the equality holds if and only if $y_1 = x_1, y_2 = x_2, \cdots, y_d = x_d$,

so

$$C_2\left(\rho^*\right) = \min_{\delta \in I} \left\|\rho - \delta\right\|_2 = C_{l_2}\left(\rho^*\right) = \sqrt{d(d-1)}\left|a\right|$$

and the minimum attains at $\delta = \rho^*_{diag}$. □

Proposition 2. For the d-dimensional maximally coherent state $\rho = |\psi_d\rangle\langle\psi_d|$, where $|\psi_d\rangle = \frac{1}{\sqrt{d}}\sum_{n=0}^{d-1}|n\rangle, d \ge 1$, we have

$$C_2(\rho) = \frac{\sqrt{d(d-1)}}{d}$$

and its optimal incoherent states is $\delta = \rho_{diag}$.

Proof. Denote the qudit incoherent state as $\delta = \sum_{k=i}^{d-1}\delta_{ii}|i\rangle\langle i|$. Let

$$U_n = \sum_{k=0}^{d-1}|k \oplus_d n\rangle\langle k|, \quad n = 0,1,2,\ldots,d-1,$$

then for $n = 0,1,2,\ldots,d-1$, we have

$$U_n|\psi_d\rangle = =|\psi_d\rangle .$$

and

$$\sum_{n=0}^{d-1} U_n \delta U_n^{\dagger} = = I_d .$$

Since for all the same dimensional matrices A,B and unitary operators U, there are the following inequalities [7]

$$\left\|UAU^{\dagger}\right\|_2 = \| A\|_2; \quad \| A+B\|_2 \le \| A\|_2 + \| B\|_2 ,$$

then we have

$$\left\| |\psi_d\rangle\langle\psi_d| - I_d \right\| \ge = \left\| |\psi_d\rangle\langle\psi_d| - \frac{1}{d} I_d \right\|_2 .$$

The above proof process is similar to the proof process of Proposition 3 in [8].

So

$$C_2(\rho) = \min_{\delta\in I} \left\| |\psi_d\rangle\langle\psi_d| - \delta \right\|_2 = \left\| |\psi_d\rangle\langle\psi_d| - \frac{1}{d} I_d \right\| = \frac{\sqrt{d(d-1)}}{d} ,$$

and the minimum attains at the optimal incoherent state $\delta = \rho_{diag} = \frac{1}{d} I_d$.

3 The Schatten-2 Norms of Coherence is not a legitimate measure of coherence

Proposition 3. $C_2(\rho)$ is not a legitimate measure of coherence.

Proof. We only need to prove that $C_2(\rho)$ violates the condition (C3) of YZX framework.

Consider a special state,

$$\rho = \frac{1}{2}\rho_1 \oplus \frac{1}{2}\rho_2 ,$$

where $\rho_1 = \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|)$ and $\rho_2 = \frac{1}{3}(|3\rangle + |4\rangle + |5\rangle)(\langle 3| + \langle 4| + \langle 5|)$.

It is easy to see that

$$C_2(\rho) = \min_{\delta\in I} \| \rho - \delta\|_2 \le \|\rho - \delta_0\|_2 = \frac{\sqrt{2}}{2} ,$$

where $\delta_0 = \mathrm{diag}\left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right)$.

On the other hand, from Proposition 2 we have

$$C_2(\rho_1) = \frac{\sqrt{2}}{2}; \quad C_2(\rho_2) = \frac{\sqrt{6}}{3} ,$$

then

$$\frac{1}{2}C_2(\rho_1)+\frac{1}{2}C_2(\rho_2)=\frac{3\sqrt{2}+2\sqrt{6}}{6}.$$

It means that

$$C_2\left(\frac{1}{2}\rho_1\oplus\frac{1}{2}\rho_2\right)\neq\frac{1}{2}C_2(\rho_1)+\frac{1}{2}C_2(\rho_2).$$

Therefore, the Schatten-2 norm of coherence Schatten-2 violates the condition (C3) of YZX framework, namely, it is a illegitimate coherence measure.

4 Conclusion

We have calculated the Schatten-2 norms of coherence $C_2(\rho)$ for some states, and got the analytical formula $C_2(\rho)=C_{l_2}(\rho)=\sqrt{2}|b|$ for qubit states $\rho=\begin{pmatrix} a & b \\ b^* & 1-a \end{pmatrix}$; $C_2(\rho_X)=C_{l_2}(\rho_X)=\sqrt{2}|a_{13}|$, $C_2(\rho_Y)=C_{l_2}(\rho_Y)=\sqrt{2}|a_{12}|$, $C_2(\rho_Z)=C_{l_2}(\rho_Z)=\sqrt{2}|a_{23}|$ for three qutrit states $\rho_X=\begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{13}^* & 0 & a_{33} \end{pmatrix}$, $\rho_Y=\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{12}^* & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$, $\rho_Z=\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{23}^* & a_{33} \end{pmatrix}$; $C_2(\rho^*)=C_{l_2}(\rho^*)=\sqrt{d(d-1)}|a|$ for a class of qudit states $\rho^*=\begin{pmatrix} x_1 & a & \cdots & a \\ a & x_2 & \cdots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \cdots & x_d \end{pmatrix}$. We have also proved that their optimal incoherent states under Schatten-2 norm of coherence are all their diagonal states. Then we have calculated that $C_2(\rho)=\frac{\sqrt{d(d-1)}}{d}$ for maximally coherent states. Finally, we have proved that the Schatten-2 norm of coherence is not a legitimate coherence measure using YZX framework.

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