UNEXTENDIBLE MAXIMALLY ENTANGLED BASIS IN $C^8 \otimes C^8$*

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ABSTRACT

In order to construct unextendible maximally entangled basis (UMEB) in $C^8 \otimes C^8$, we find that it can be constructed from that in $C^4 \otimes C^4$. In this paper, we will use the 12 member UMEB in $C^4 \otimes C^4$ to construct 56-member UMEB in $C^8 \otimes C^8$, and give the corresponding proof.

I INTRODUCTION

S.Bravyi and J.A.Smolin [1] generalized the notion of the unextendible product base (UPB) to the unextendible entangled basis (UEB), which is a set of bipartite pure states $|\psi_i\rangle$ each of which has entanglement α but whose complement space is non-empty and contains no states of entanglement α. When $\alpha = 1$, the basis is called unextendible maximally entangled basis (UMEB) [1]. They also proved that there are no UMEB in $C^2 \otimes C^2$ and constructed a 6-member UMEB in $C^3 \otimes C^3$ and a 12-member UMEB in $C^4 \otimes C^4$.

Bin Chen and Shao-Ming Fei [2] provided a systematic ways of constructing a set of d^2 orthonormal maximally entangled states in $C^d \otimes C^{d'}$ $\left(\frac{d'}{2} < d < d'\right)$. In 2014, Mao-Sheng Li and Yan-Ling Wang [3] gave an explicit construction of UMEB by considering the Schmidt number of the complementary space of the states they constructed. Later, they [4] showed that for a give N-member UMEB in $C^d \otimes C^d$, there is a $\tilde{N}$-member UMEB in $C^{qd} \otimes C^{qd}$ for any $q \in \mathrm{N}$ where $\tilde{N} = (qd)^2 - q(d^2 - N)$.

In this paper, we try to contruct maximally entangled states in $C^8 \otimes C^8$ from that in $C^4 \otimes C^4$, and prove that these states constitute a UMEB.

Ⅱ.UMEBS IN $\mathbf{C^{qd} \otimes C^{qd}}$

Definition[1] A set of states $\{|\psi_i\rangle \in C^d \otimes C^d : a = 1,2,\dots,\mathrm{n},\ \mathrm{n} < \mathrm{d}^2\}$ is called an n-member UMEB if and only if:

(i) $|\psi_a\rangle, a = 1,2,\dots,\mathrm{n}$, are maximally entangled;

(ii) $\langle \Phi_a | \Psi \rangle = \delta_{\mathrm{ab}}$;

(iii) If $\langle \Phi_a | \Psi \rangle = 0$ for all $\mathrm{a} = 1,2,\dots,\mathrm{n}$, then $|\Psi\rangle$ cannot be maximally entangled.

Here, $|\Phi_a\rangle$ can be expressed as

$$|\Phi_a\rangle = (I \otimes U_a)\frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|i\rangle \otimes |i\rangle, \quad (1)$$

where, I is the $\mathrm{d} \times \mathrm{d}$ identity maxtrix, $\mathrm{U_a}$ is any unitary maxtrix. We can know form (1) that,

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the conditions (i-iii) can now be rephrased as [1]

(i) U_a, a = 1, 2, …, n, and all unitary metrices;

(ii) $\mathrm{Tr}(U_a{}^{\dagger}U_b) = d\delta_{ab}$;

(iii) If $\mathrm{Tr}(U_a{}^{\dagger}U) = 0$ for all $a = 1,2,..,$n, then U cannot be unitary.

Lemma[4] If there is an N-member UMEB in $C^d \otimes C^d$, then for any $q \in$ N, there is a $\tilde{N}$-member, $\tilde{N} = (qd)^2 - q(d^2 - N)$, UMEB in $C^{qd} \otimes C^{qd}$.

Let $\{U_n\}, n = 1,2,..,N < d^2$, be the set of unitary matrices that given rise to the N-member UMEB in $C^d \otimes C^d$, and

$$S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi_q & \xi_q^2 & \cdots & \xi_q^{q-1} \\ 1 & \xi_q^2 & \xi_q^4 & \cdots & \xi_q^{2(q-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_q^{q-1} & \xi_q^{2(q-1)} & \cdots & \xi_q^{(q-1)^2} \end{bmatrix},$$

where $\xi_q = e^{\frac{2\pi\sqrt{-1}}{q}}$.

Denote $U_{nm} = \sum_{k=0}^{d-1} e^{\frac{2\pi\sqrt{-1}}{q}kn}|k \oplus m\rangle\langle k|$. m ,n = 0, …, d-1,

Set

$$U_{nm}^{gh} = (W^g S^h) \otimes U_{nm}, \quad 0 \le g \le q-1, 1 \le h \le q-1, m, n = 0, \dots, d-1. \tag{2}$$

$$U_n^g = W^g \otimes U_n, \quad g = 0,1, \dots, q-1, \; n = 1,2, \dots, N < d^2 \tag{3}$$

then $\{U_{nm}^{gh}, U_n^g\}$ give a $(qd)^2 - q(d^2 - N)$-member UMEB in $C^{qd} \otimes C^{qd}$.

The specific construction method will be described in detail by the example of UMEB in $C^8 \otimes C^8$.

Ⅲ.UMEBS IN $C^8 \otimes C^8$

According to the above content, we can construct UMEB in $C^8 \otimes C^8$ from the UMEB in $C^2 \otimes C^2$ or in $C^4 \otimes C^4$ because of $8 = 2 \times 4$. In Ref [1], we have known that UMEBs do not exist in $C^2 \otimes C^2$. Therefore, in order to construct UMEBs in $C^8 \otimes C^8$, we have to use the 12-member UMEBs in $C^4 \otimes C^4$.

We can calculate that there are 56-member UMEBs in $C^8 \otimes C^8$ by the formula $\tilde{N} = (qd)^2 - q(d^2 - N)$, where $q = 2, d = 4, N = 12$. Now, it's time to construct the 56-member UMEBs by two parts.

1. 32 maximally entangled states in $\{U_{nm}^{gh}\}$

In this case, it's easy to know that:

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

From the formula (2), we can get that

$$U_{nm}^{gh} = (W^g S^h) \otimes U_{nm},$$

where

$$W^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad W^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad w = e^{\frac{2\pi\sqrt{-1}}{4}} = i, \quad S^h = S, \quad h = 1,$$

$$U_{nm} = \sum_{k=0}^{3} e^{(-1)\times kn}|k \oplus m\rangle\langle k|, \quad m, n = 0,12,3.$$

With the formula (1), we get the following 32 maximally entangled states

$$|\Phi_a\rangle = \left(I\otimes U_{nm}^{gh}\right)\frac{1}{2\sqrt{2}}\sum_{g=0}^{7}|g\rangle\otimes|g\rangle = \frac{1}{2\sqrt{2}}\sum_{g=0}^{7}|g\rangle\otimes\left(\left((W^g S^h)\otimes U_{nm}\right)|g\rangle\right), \quad (8)$$

where r=1,2, s=1, m,n=0,1,2,3

The above 32 maximally entangled states in (8) are as follows:

$$|\Phi_{1,2,\dots,8}\rangle = \frac{1}{2\sqrt{2}}(a_1|04'\rangle + a_2|15'\rangle + a_3|26'\rangle + a_4|37'\rangle + a_5|40'\rangle + a_6|51'\rangle + a_7|62'\rangle + a_8|73'\rangle),$$

$$|\Phi_{9,10,\dots,16}\rangle = \frac{1}{2\sqrt{2}}(a_1|05'\rangle + a_2|16'\rangle + a_3|27'\rangle + a_4|34'\rangle + a_5|41'\rangle + a_6|52'\rangle + a_7|63'\rangle + a_8|74'\rangle),$$

$$|\Phi_{17,18,\dots,24}\rangle = \frac{1}{2\sqrt{2}}(a_1|06'\rangle + a_2|17'\rangle + a_3|24'\rangle + a_4|35'\rangle + a_5|42'\rangle + a_6|53'\rangle + a_7|64'\rangle + a_8|75'\rangle),$$

$$|\Phi_{25,26,\dots,32}\rangle = \frac{1}{2\sqrt{2}}(a_1|07'\rangle + a_2|14'\rangle + a_3|25'\rangle + a_4|36'\rangle + a_5|43'\rangle + a_6|53'\rangle + a_7|60'\rangle + a_8|71'\rangle),$$

the above coefficients $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ take eight column values in the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i & 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i & 1 & -i & -1 & i \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & i & -1 & -i & -1 & -i & 1 & i \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -i & -1 & i & -1 & i & 1 & -i \end{bmatrix}.$$

2. 24 maximally entangled states in $\{U_n^g\}$

In Ref. [1], we get the following 12- member UMEBs in $C^4\otimes C^4$ explicitly:

$$U_1 = \frac{1}{\sqrt{2}}\sigma_x\otimes(\sigma_x - \sigma_y),\ U_2 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y)\otimes\sigma_z,\ U_3 = \frac{1}{\sqrt{2}}\sigma_z\otimes(-\sigma_y + \sigma_z),$$

$$U_4 = \frac{1}{\sqrt{2}}(-\sigma_y + \sigma_z)\otimes\sigma_z,\ U_5 = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)\otimes(\sigma_x + \sigma_y + \sigma_z),$$

and $\{U_6, \dots, U_{12}\} = \{I\otimes I, I\otimes\sigma_x, I\otimes\sigma_y, I\otimes\sigma_z, \sigma_x\otimes I, \sigma_y\otimes I, \sigma_z\otimes I\}$, where $\sigma_x, \sigma_y, \sigma_z$ are the Pauli matrices.

From the formula (3), we get that:

$$U_n^1 = \begin{bmatrix} U_n & O_4 \\ O_4 & U_n \end{bmatrix},\ U_n^2 = \begin{bmatrix} U_n & O_4 \\ O_4 & -U_n \end{bmatrix},$$

where $n = 1,2,\dots,12$.

By $|\psi_g\rangle = (I\otimes U_a)\frac{1}{2\sqrt{2}}\sum_{g=0}^{7}|g\rangle\otimes|g\rangle$, we can get the other 24 maximally entangled states.

Next, we prove that the above 56 maximally entangled states in $\{U_{nm}^{gh}, U_n^g\}$ is a 56-member UMEB in $C^8\otimes C^8$.

(1) Since W^g, S, U_{nm} are all unitary, so are $\{U_{nm}^{gh}, U_n^g\}$. It's obvious to find that $n = 56 < d^2 = 64$, so the given set of matrices satisty the first condition of UMEB.

(2) In order to prove the orthogonality of the these unitary states, we consider three different cases:

(i) inner products in between two elements in $\{U_{nm}^{gh}\}$

$$\operatorname{Tr}\left(\left(U_{nm}^{gh}\right)^{\dagger}U_{\tilde{n}\tilde{m}}^{\tilde{g}\tilde{h}}\right)=\operatorname{Tr}\left(\left((W^{g}S^{h})\otimes U_{nm}\right)^{\dagger}\left((W^{\tilde{g}}S^{\tilde{h}})\otimes U_{\tilde{n}\tilde{m}}\right)\right)$$

$$=\operatorname{Tr}\left(\left((W^{g}S^{h})^{\dagger}(W^{\tilde{g}}S^{\tilde{h}})\right)\otimes U_{nm}^{\dagger}U_{\tilde{n}\tilde{m}}\right)$$

$$=8\delta_{g\tilde{g}}\delta_{h\tilde{h}}\delta_{n\tilde{n}}\delta_{m\tilde{m}}.$$

(ii) inner products in between two elements in $\{U_{n}^{g}\}$

$$\operatorname{Tr}\left(\left(U_{n}^{g}\right)^{\dagger}U_{\tilde{n}}^{\tilde{h}}\right)=\operatorname{Tr}\left((W^{g}\otimes U_{nm})^{\dagger}(W^{g}\otimes U_{\tilde{n}})\right)$$

$$=\operatorname{Tr}\left(\left((W^{g})^{\dagger}(W^{\tilde{h}})\right)\otimes U_{n}^{\dagger}U_{\tilde{n}}\right)$$

$$=8\delta_{g\tilde{g}}\delta_{n\tilde{n}}.$$

(iii) inner products between one elements in $\{U_{nm}^{gh}\}$ and the other one in $\{U_{n}^{g}\}$

$$\operatorname{Tr}\left(\left(U_{nm}^{gh}\right)^{\dagger}U_{\tilde{n}}^{\tilde{g}}\right)=\operatorname{Tr}\left(\left((W^{g}S^{h})\otimes U_{nm}\right)^{\dagger}(W^{\tilde{g}}\otimes U_{\tilde{n}})\right)$$

$$=\operatorname{Tr}\left(\left((W^{g}S^{h})^{\dagger}W^{\tilde{g}}\right)\otimes U_{nm}^{\dagger}U_{\tilde{n}}\right)$$

$$=0.$$

So the given set of matrices satisty the second condition of UMEB.

(3) Assume that $U\in \mathrm{M}_8(\mathrm{C})$ satisty

$$\operatorname{Tr}\left(U^{\dagger}U_{nm}^{gh}\right)=0 \text{ and } \operatorname{Tr}\left(U^{\dagger}U_{n}^{g}\right)=0,$$

Let $\mathrm{V}_1=\operatorname{span}\{U_{nm}^{gh}\}$, $\dim\mathrm{V}_1=32$. Denote

$$\mathrm{V}_2=\left\{\begin{bmatrix}\mathrm{A}_1 & 0\\ 0 & \mathrm{A}_2\end{bmatrix}\middle|\mathrm{A}_1,\mathrm{A}_2\in \mathrm{M}_4(\mathrm{C})\right\}$$

then $\dim\mathrm{V}_2=32$. We have $\operatorname{Tr}\left({A_q}^{\dagger}U_{nm}^{gh}\right)=0$ $(\mathrm{q}=1,2)$. Thus for any matrix $B\in\mathrm{V}_1$, $\operatorname{Tr}\left({A_q}^{\dagger}B\right)=0$ $(\mathrm{q}=1,2)$. Namely, $\mathrm{V}_2\subseteq\mathrm{V}_1^{\perp}$. Accounting to the dimensions of $\mathrm{V}_1,\mathrm{V}_2$ and $\mathrm{M}_8(\mathrm{C})$, we obtain ${\mathrm{V}_1}^{\perp}=\mathrm{V}_2$. Set $\mathrm{V}_3=\operatorname{span}\{U_{nm}^{gh},U_{n}^{g}\}$. Clearly, ${\mathrm{V}_3}^{\perp}\subset{\mathrm{V}_1}^{\perp}=\mathrm{V}_2$. Therefore $U\in{\mathrm{V}_3}^{\perp}$, and the matrix U has the form

$$U=\operatorname{diag}(\mathrm{W}_1,\mathrm{W}_2) \quad \text{where } \mathrm{W}_1,\mathrm{W}_2\in \mathrm{M}_4(\mathrm{C})$$

In addition, $\operatorname{Tr}=\left(U^{\dagger}U_{n}^{g}\right)=0$, we have

$$\operatorname{Tr}\left(\begin{bmatrix}\mathrm{W}_1 & 0\\ 0 & \mathrm{W}_2\end{bmatrix}^{\dagger}\begin{bmatrix}U_g & 0\\ 0 & \pm U_g\end{bmatrix}\right)=0$$

i.e. $\operatorname{Tr}\left({\mathrm{W}_1}^{\dagger}U_g\right)\pm\operatorname{Tr}\left({\mathrm{W}_2}^{\dagger}U_g\right)=0$, we have $\operatorname{Tr}\left({\mathrm{W}_1}^{\dagger}U_g\right)=\operatorname{Tr}\left({\mathrm{W}_2}^{\dagger}U_g\right)=0$ for $g=1,\dots 8$, which implies $\mathrm{W}_1,\mathrm{W}_2\notin \mathrm{U}(4)$ and hence $U\notin U(8)$. Therefore we conclude that $\{U_{nm}^{gh},U_{n}^{g}\}$ is a 56-member UMEB in $C^8\otimes C^8$.

Ⅳ.CONCLUSION

We have constructed 32 and 24 maximally entangled states from $\{U_{nm}^{gh}\}$ and $\{U_{n}^{g}\}$, respectively. Then we proved their orthogonality in three cases: inner products between two elements in $\{U_{nm}^{gh}\}$, inner products between two elements in $\{U_{n}^{g}\}$, inner products between one elements in $\{U_{nm}^{gh}\}$ and the other one in $\{U_{n}^{g}\}$. We can calculate that there are 56 members UMEB. In the end, we verify the conclusion of the literature [4] with the structure of UMEB in $C^8 \otimes C^8$.

V. REFERENCE

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