WEAK ONE-WAY DEFICIT OF THE ISOTROPIC STATE

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Abstract Isotropic state is a simple case of Bell-diagonal states. We first calculate its weak one-way deficit, then we make comparisons among its quantum discord, one-way deficit, super-quantum discord and weak one-way deficit. We find that for the isotropic state, the quantum discord is equal to the one-way deficit, and the super-quantum discord is larger than the quantum discord while the weak one-way deficit is less than the one-way deficit.

Keywords quantum discord; one-way deficit; super-quantum discord; weak one-way deficit; isotropic state

I. Introduction

Quantum discord, a measure of correlation, was introduced by by Ollivier and Zurek [1]. It quantifies the difference between the quantum mutual information and the maximal classical information, and can exist in separable mixed quantum states [2,3]. The quantum discord for a bipartite quantum state ρ^{ab} on subsystemB is defined as [4]

$$D(\rho^{ab}) = \min_{\{\Pi_i^B\}} \sum_i p_i S(\rho_{A|i}) + S(\rho^b) - S(\rho^{ab}) . \tag{1}$$

where the minimum is among all projective measurement $\{\Pi_i^B\}$, $S(\rho) = -tr(\rho \log_2 \rho)$ is the von Neumann entropy , ρ^b is the reduced density matrix of ρ^{ab} and

$$p_i = tr_{AB}[(I_A \otimes \Pi_i^B)\rho_{AB}(I_A \otimes \Pi_i^B)] , \tag{2}$$

$$\rho_{A|i} = \frac{1}{p_i} tr_{AB}[(I_A \otimes \Pi_i^B)\rho_{AB}(I_A \otimes \Pi_i^B)] . \tag{3}$$

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Weak measurement [5] was firstly proposed by Aharonov, Albert, and Vaidman in 1988. It plays an important role in interpreting counterintuitive quantum phenomena such as Hardy's paradoxes [6–8]. The weak measurement operators are given by [9]

$$P(x)=\sqrt{\frac{(1-\tanh x)}{2}}\Pi_0+\sqrt{\frac{(1+\tanh x)}{2}}\Pi_1,$$
$$P(-x)=\sqrt{\frac{(1+\tanh x)}{2}}\Pi_0+\sqrt{\frac{(1-\tanh x)}{2}}\Pi_1, \tag{4}$$

where x means the measurement strength parameter, Π_0 and Π_1 are two orthogonal projectors and $\Pi_0+\Pi_1=\mathrm{I}$. The weak measurement operators have two properties: (i) $P^{\dagger}(x)P(x)+P^{\dagger}(-x)P(-x)=I$; (ii) $\lim_{x\to\infty}P(x)=\Pi_0$ and $\lim_{x\to\infty}P(-x)=\Pi_1$.

Super-quantum discord [3] is identified as quantum discord preformed by weak measurements. The super-quantum discord [3] for bipartite quantum state ρ^{ab} measured locally on the B system with weak measurement was defined as follows

$$D_w(\rho^{ab})=\min_{\{\Pi_i^B\}}S_w(\mathrm{A}\,|\,\{\mathrm{P}(x)\})+S(\rho^b)-S(\rho^{ab}), \tag{5}$$

where

$$\min_{\{\Pi_i^B\}}S_w(\mathrm{A}\,|\,\{\mathrm{P}(x)\})=p(x)\mathrm{S}(\rho_{A|P^B(x)})+p(-x)\mathrm{S}(\rho_{A|P^B(-x)}), \tag{6}$$

$$p(\pm x)=tr_{AB}[(I\otimes P^B(\pm x))\rho_{AB}(I\otimes P^B(\pm x))], \tag{7}$$

$$\rho_{A|P^B(\pm x)}=\frac{tr_B[(I\otimes P^B(\pm x))\rho_{AB}(I\otimes P^B(\pm x))]}{p(\pm x)}. \tag{8}$$

Quantum deficit [10] is another measure of correlation except discord, which was defined by Streltsov et al. The one-way deficit, the difference between the mutual information and classical deficit, has the interpretation connection with discord. The one-way deficit of ρ^{ab} is given as [10]

$$\Delta^{\rightarrow}(\rho^{ab})=\min_{\{\Pi_k^B\}}S(\sum_i\Pi_k\rho^{ab}\Pi_k)-\mathrm{S}(\rho^{ab}). \tag{9}$$

Weak one-way deficit[11] is described as one way deficit via weak measurements as follows

$$\Delta_w^{\rightarrow}(\rho^{ab}) = \min_{\{\Pi_k^B\}} S\left(\sum_{P(\pm x)}\left[I\otimes P^B(\pm x)\right]\rho^{ab}\left[I\otimes P^B(\pm x)\right]\right) - S(\rho^{ab}). \qquad (10)$$

In this paper, we first show the previous conclusions about quantum discord, one-way deficit and super-quantum discord of Bell-diagonal state in section II. In section III, we show that the isotropic state is a special Bell-diagonal state and calculate the weak one-way deficit of isotropic state. In section IV, we make comparisons among the quantum discord, the one-way deficit, the super-quantum discord and the weak one-way deficit of the isotropic state.

II. Quantum discord, one-way deficit and super-quantum discord of Bell-diagonal states

The Bell-diagonal states[12–14] of two qubits are defined as follows:

$$\rho^{ab} = \frac{1}{4}(I\otimes I + \sum_i^3 c_i\sigma_i\otimes\sigma_i), \qquad (11)$$

where c_i are complex parameters, I is the identity matrix, σ_i are Pauli operators.

The quantum discord of Bell-diagonal states was given as follows by S. L. Luo in 2008[3]

$$\begin{aligned} D(\rho) = \frac{1}{4}[&(1-c_1-c_2-c_3)\log_2(1-c_1-c_2-c_3) \\ &+(1-c_1+c_2+c_3)\log_2(1-c_1+c_2+c_3) \\ &+(1+c_1-c_2+c_3)\log_2(1+c_1-c_2+c_3) \\ &+(1+c_1+c_2-c_3)\log_2(1+c_1+c_2-c_3)] \\ &-\frac{1-c}{2}\log_2(1-c)-\frac{1+c}{2}\log_2(1+c) \end{aligned} \qquad (12)$$

where $c=\max\{|c_1|,|c_2|,|c_3|\}$.

The one-way deficit of Bell-diagonal $\Delta^{\rightarrow}(\rho^{ab})$ was evaluated in [15], which is equal to its quantum discord in (12), i.e., $D(\rho^{ab})=\Delta^{\rightarrow}(\rho^{ab})$. The super-quantum discord for Bell-diagonal state was calculated in [16], that is :

$$
\begin{aligned}
D_w\left(\rho^{ab}\right) = & -\frac{1-c\tanh x}{2}\log_2\frac{1-c\tanh x}{2} - \frac{1+c\tanh x}{2}\log_2\frac{1+c\tanh x}{2} \\
& +1+\frac{1-c_1-c_2-c_3}{4}\log_2\frac{1-c_1-c_2-c_3}{4} \\
& +\frac{1-c_1+c_2+c_3}{4}\log_2\frac{1-c_1+c_2+c_3}{4} \\
& +\frac{1+c_1-c_2+c_3}{4}\log_2\frac{1+c_1-c_2+c_3}{4} \\
& +\frac{1+c_1+c_2-c_3}{4}\log_2\frac{1+c_1+c_2-c_3}{4} .
\end{aligned} \tag{13}
$$

III. Weak one-way deficit of the isotropic state

The $m \times m$ dimensional isotropic state was given by Horodecki [18]:

$$\rho_x = \frac{1-x}{m^2-1} I + \frac{m^2 x - 1}{m^2-1}|\psi\rangle\langle\psi|, \qquad x \in [0, 1]. \tag{14}$$

where $|\psi\rangle = \frac{1}{\sqrt{m}}\sum_{k=1}^{m}|k\rangle\otimes|k\rangle$. Then in 2×2 quantum system, we have the density operator of isotropic state as follows

$$\rho^{ab} = \frac{I}{4} + \frac{4a-1}{12}(\sigma_1\otimes\sigma_1 - \sigma_2\otimes\sigma_2 + \sigma_3\otimes\sigma_3), \tag{15}$$

where the real parameter $a \in [\frac{1}{2}, 1]$. It is easy to verify that the isotropic state is a special Bell-diagonal state in (11) with $c_1 = \frac{4a-1}{3}$, $c_2 = -\frac{4a-1}{3}$ and $c_3 = \frac{4a-1}{3}$. Actually, the density operator of isotropic state in (15) can be written as follows

$$\rho^{ab} = \begin{pmatrix} 2a+1 & 0 & 0 & 4a-1 \\ 0 & 2(1-a) & 0 & 0 \\ 0 & 0 & 2(1-a) & 0 \\ 4a-1 & 0 & 0 & 2a+1 \end{pmatrix} \tag{16}$$

It is easy to calculate the eigenvalues of the isotropic state in (16), that is

$$\lambda_1 = a, \quad \lambda_{2,3,4} = \frac{a-1}{3}, \tag{17}$$

then the Von Neumann entropy of the isotropic state ρ^{ab} is

$$S(\rho^{ab}) = -\sum_i^4 \lambda_i \log_2 \lambda_i = -(1-a)\log_2(\frac{1-a}{3}) - a\log_2 a . \tag{18}$$

Denote the post-measurement isotropic state as ρ', then

$$\rho' = \sum_{P(\pm x)} \left[I \otimes P^B(\pm x) \right] \rho^{ab} \left[I \otimes P^B(\pm x) \right] \tag{19}$$

Since the isotropic state is a local unitary equivalence class, i.e.,

$$(U \otimes V)\rho^{ab}(U \otimes V)^{\dagger} = \rho^{ab}, \tag{20}$$

where U and V are arbitrary unitary operators on subsystem A and B respectively, then

$$\rho' = \sum_{P(\pm x)} \left[I \otimes P^B(\pm x) \right](U \otimes V)\rho^{ab}(U \otimes V)^{\dagger}\left[I \otimes P^B(\pm x) \right], \tag{21}$$

then the Von Neumann entropy of ρ' is

$$\begin{aligned} S(\rho') &= S\left(\sum_{P(\pm x)} \left[I \otimes P^B(\pm x) \right](U \otimes V)\rho^{ab}\left(U^{\dagger} \otimes V^{\dagger}\right)\left[I \otimes P^B(\pm x) \right] \right) \\ &= S\left(\sum_{P(\pm x)} \left[U \otimes P^B(\pm x)V \right]\rho^{ab}\left[U^{\dagger} \otimes V^{\dagger}P^B(\pm x) \right] \right) \\ &= S\left(\left(U^{\dagger} \otimes V^{\dagger}\right)\left[\sum_{P(\pm x)} \left(U \otimes P^B(\pm x)V\right)\rho^{ab}\left(U^{\dagger} \otimes V^{\dagger}P^B(\pm x)\right) \right](U \otimes V) \right) \\ &= S\left(\sum_{P(\pm x)} \left[I \otimes V^{\dagger}P^B(\pm x)V \right]\rho^{ab}\left[I \otimes V^{\dagger}P^B(\pm x)V \right] \right), \end{aligned} \tag{22}$$

thus the Von Neumann entropy of post-measurement isotropic state ρ' has no change under any weak measurement on subsystem B, therefore

$$\min_{\{\Pi_k\}} S(\rho') = S(\rho'). \tag{23}$$

Substituting (7) and (15) into (19), then we have

$$\begin{aligned} \rho' &= \sum_{P(\pm x)} \left[I \otimes \left(\sqrt{\frac{(1 \mp \tanh x)}{2}} U^{\dagger}\Pi_0 U + \sqrt{\frac{(1 \pm \tanh x)}{2}} U^{\dagger}\Pi_1 U \right) \right] \cdot \rho^{ab} \\ &\quad \cdot \left[I \otimes \left(\sqrt{\frac{(1 \mp \tanh x)}{2}} U^{\dagger}\Pi_0 U + \sqrt{\frac{(1 \pm \tanh x)}{2}} U^{\dagger}\Pi_1 U \right) \right] \\ &= \sum_{P(\pm x)} \left[I \otimes \left(\sqrt{\frac{1 \mp \tanh x}{2}} |0\rangle\langle 0| + \sqrt{\frac{1 \pm \tanh x}{2}} |1\rangle\langle 1| \right) \right] \cdot \left[\frac{(1-a)}{3} I + \frac{4a-1}{6}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right) \right] \end{aligned} \tag{24}$$

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$$\cdot\left[I\otimes\left(\sqrt{\frac{1\mp\tanh x}{2}}|0\rangle\langle 0|+\sqrt{\frac{1\pm\tanh x}{2}}|1\rangle\langle 1|\right)\right]$$

$$=\frac{2a+1}{6}|00\rangle\langle 00|+\frac{1-a}{3}|01\rangle\langle 01|+\frac{1-a}{3}|10\rangle\langle 10|+\frac{4a-1}{6\cosh x}|00\rangle\langle 11|+\frac{4a-1}{6\cosh x}|11\rangle\langle 00|$$
$$+\frac{2a+1}{6}|11\rangle\langle 11|,$$

then the eigenvalues of ρ' are as follows,

$$\lambda_5=\frac{2a+1}{6}+\frac{4a-1}{6\cosh x},\lambda_6=\frac{2a+1}{6}-\frac{4a-1}{6\cosh x},\lambda_{7,8}=\frac{1-a}{3}, \tag{25}$$

then,

$$\begin{aligned}S(\rho')=&\frac{2a+1}{3}-\left(\frac{2a+1}{6}+\frac{4a-1}{6\cosh x}\right)\log_2\left(\frac{2a+1}{3}+\frac{4a-1}{3\cosh x}\right)\\&-\left(\frac{2a+1}{6}+\frac{4a-1}{6\cosh x}\right)\log_2\left(\frac{2a+1}{3}-\frac{4a-1}{3\cosh x}\right)-\frac{2(1-a)}{3}\log_2\frac{1-a}{3},\end{aligned} \tag{26}$$

hence the weak one-way deficit $\Delta_w^{\rightarrow}\left(\rho^{ab}\right)$ of the isotropic state is

$$\begin{aligned}\Delta_w^{\rightarrow}\left(\rho^{ab}\right)=&\frac{2a+1}{3}+\frac{1-a}{3}\log_2\frac{1-a}{3}+a\log_2 a\\&-\left(\frac{2a+1}{6}+\frac{4a-1}{6\cosh x}\right)\log_2\left(\frac{2a+1}{3}+\frac{4a-1}{3\cosh x}\right)\\&-\left(\frac{2a+1}{6}-\frac{4a-1}{6\cosh x}\right)\log_2\left(\frac{2a+1}{3}-\frac{4a-1}{3\cosh x}\right)\end{aligned} \tag{27}$$

IV. Comparisons

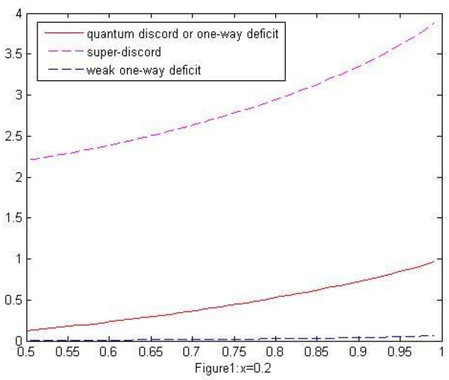
Since the isotropic state is a special Bell-diagonal state with $c_1=\frac{4a-1}{3}$, $c_2=-\frac{4a-1}{3}$ and $c_3=\frac{4a-1}{3}$. Then from (12), we get the quantum discord $D\left(\rho^{ab}\right)$ and one-way deficit $\Delta^{\rightarrow}\left(\rho^{ab}\right)$ of isotropic state as follows

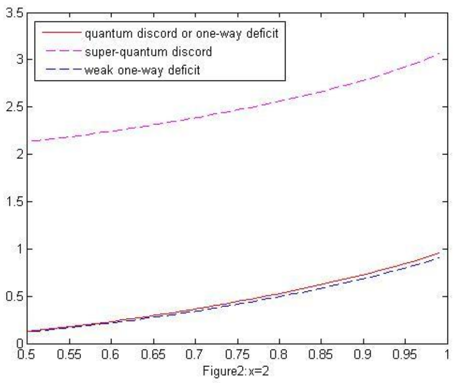
$$D(\rho^{ab})=\Delta^{\rightarrow}(\rho^{ab})=\frac{1+2a}{3}+\frac{1-a}{3}\log_2(\frac{1-a}{3})+a\log_2 a-\frac{1+2a}{3}\log_2(\frac{1+2a}{3}). \tag{29}$$

According to (13), the super-quantum discord $D_w\left(\rho^{ab}\right)$ of the isotropic state is

$$
\begin{aligned}
D_w\left(\rho^{ab}\right) = 4 + (1-a)\log_2\frac{1-a}{3} + a\log_2 a \\
-\frac{3-(4a-1)\tanh x}{6}\log_2\left(1-\frac{4a-1}{3}\tanh x\right) \\
-\frac{3+(4a-1)\tanh x}{6}\log_2\left(1+\frac{4a-1}{3}\tanh x\right).
\end{aligned}
\tag{30}
$$

Then, we plot the graphs of the quantum discord $D\left(\rho^{ab}\right)$(the one-way deficit $\Delta^{\rightarrow}\left(\rho^{ab}\right)$), the super-quantum discord $D_w\left(\rho^{ab}\right)$ and the weak one-way deficit $\Delta_w^{\rightarrow}\left(\rho^{ab}\right)$ of the isotropic state in Figure 1 and Figure 2 with the parameter x=0.2 and x=2 respectively. From the figures we can find that the super-quantum discord $D_w\left(\rho^{ab}\right)$ is greater than the quantum discord $D\left(\rho^{ab}\right)$ (the one-way deficit $\Delta^{\rightarrow}\left(\rho^{ab}\right)$) while the weak one-way deficit $\Delta_w^{\rightarrow}\left(\rho^{ab}\right)$ is smaller than the one-way deficit $\Delta^{\rightarrow}\left(\rho^{ab}\right)$, that is, $\Delta_w^{\rightarrow}\left(\rho^{ab}\right) < \Delta^{\rightarrow}\left(\rho^{ab}\right) = D\left(\rho^{ab}\right) < D_w\left(\rho^{ab}\right)$. Meanwhile, the weak one-way deficit $D_w\left(\rho^{ab}\right)$ tends to zero along the decrease of x. along the increase of x tht weak one-way deficit $\Delta_w^{\rightarrow}\left(\rho^{ab}\right)$ increases and approaches to the quantum discord $D\left(\rho^{ab}\right)$ (the one-way deficit $\Delta^{\rightarrow}\left(\rho^{ab}\right)$).





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