Unsaturated Solutions of A Nonlinear Delay Partial Difference Equation with Variable Coefficients

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Abstract: By employing the concept and the properties of frequency measures of infinite double sequence and the concept of unsaturated solution of partial difference equations, the unsaturated solutions for the nonlinear partial difference equation with variable coefficients is discussed. Only using the concept of "frequency measure" of the level sets of the involved parameter sequences in equation, the sufficient conditions of the solutions to be unsaturated are presented, thus how frequent the solutions oscillate is well described.

Keywords: partial difference equation; frequency measure; unsaturated solution

1. Introduction

In recent years, many results have been found about the oscillatory and Non-oscillatory solutions of difference equations. However, the classical concept of oscillation still can not describe the oscillation of sequences. Therefore, Chuanjun Tian [1] first introduced the concept of frequency measure of sequence, and described frequent oscillation of sequence. In order to further improve the frequent oscillation of sequence, Zhiqiang Zhu et al. [2] also defined the concept of frequently positive oscillation and frequently negative oscillation of sequence. At present, there are some results about the frequent oscillation of solutions of difference equations, see [3-9].

Let Z be the set of integers, $Z[k, l]=\{i \in Z | i=k, k+1, \cdots, l\}$ and

$Z[k,\infty)=\{i \in Z | i=k,k+1,\cdots\}$.

In this paper, we discuss the unsaturated solutions of the following nonlinear partial difference equation with coefficients of variable sign

$$u_{m,n} = u_{m-1,n} + u_{m,n+1} + p_{m,n}|u_{m+k_1,n-l_1}|^{\alpha} \operatorname{sgn} u_{m+k_1,n-l_1} + q_{m,n}|u_{m+k_2,n-l_2}|^{\beta} \operatorname{sgn} u_{m+k_2,n-l_2}, \tag{1.1}$$

where m, $n \in Z[0,\infty)$, $\alpha \in [0,1)$, $\beta \in (1,\infty)$; k_1, k_2, l_1 and l_2 are all nonnegative integers, $k_1 > k_2 \geq 0$, $l_1 > l_2 \geq 0$. Moreover, $p = \{p_{m,n}\}_{m,n \in Z[0,\infty)}$ and $q = \{q_{m,n}\}_{m,n \in Z[0,\infty)}$ are real double sequences satisfying the condition $(*)$: if $p = \{p_{m,n}\}$ has negative item, then

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the denominator of $\frac{\beta-1}{\beta-\alpha}$ is a positive odd number; similarly, if $q=\{q_{m,n}\}$ has negative item, then the denominator of $\frac{1-\alpha}{\beta-\alpha}$ is also a positive odd number.

In the sequel, $p=\{p_{m,n}\}$ and $q=\{q_{m,n}\}$ may have negative items. Generally, for $(m,n)\in[-1,\infty)\times[-1,\infty)$, the double sequence $\{u_{m,n}\}_{(m,n)\in[-1,\infty)\times[-1,\infty)}$ is called the solution of equation (1.1) if it keeps the equation (1.1) holds.

2. Preliminary

Let $Z^2=Z\times Z$, we call an element of Z^2 to be a lattice. Denote the union, intersection and difference of two sets A and B to be $A+B$, $A\cdot B$ and $A\setminus B$ respectively. If $\Omega\subseteq Z^2$, then we denote the potential of Ω to be $|\Omega|$; and denoting $\Omega^{(m,n)}=\{(s,t)\in\Omega|s\le m,t\le n\}$.

For any $\Omega\in Z^2$ and integers m,n, set

$$X^m(\Omega)=\{(s+m,t)\in Z^2|(s,t)\in\Omega\};\quad Y^n(\Omega)=\{(s,t+n)\in Z^2|(s,t)\in\Omega\}.$$

and $X_\alpha^\beta Y_\gamma^\tau\Omega=\sum_{s=\alpha}^{\beta}\sum_{t=\gamma}^{\tau}X^sY^t\Omega$, where α,β,γ and τ are all integers satisfying $\alpha\le\beta,\gamma\le\tau$, then

$$(s,t)\in Z^2\setminus X_\alpha^\beta Y_\gamma^\tau\Omega\Leftrightarrow(s-k,t-l)\in Z^2\setminus\Omega,\quad \alpha\le k\le\beta,\gamma\le l\le\tau. \tag{2.1}$$

Definition 2.1[1] Let $\Omega\subseteq Z^2$, if the upper limit $\limsup_{m,n\to\infty}\frac{|\Omega^{(m,n)}|}{mn}$ exists, then we call the limit to be the upper frequent measure of Ω, denoting $\mu^*(\Omega)$. Similarly, if the lower limit $\liminf_{m,n\to\infty}\frac{|\Omega^{(m,n)}|}{mn}$ exists, then we call the limit to be the lower frequent measure of Ω, denoting $\mu_*(\Omega)$. If $\mu^*(\Omega)=\mu_*(\Omega)$,then the corresponding limit is called the frequent measure of Ω, denoting $\mu(\Omega)$.

Definition 2.2[1] Let $u=\{u_{m,n}\}_{m,n=1}^{\infty}$ be an arbitrary real double sequence, if $\mu^*(u\le 0)=0$, then we say that u is frequently positive, if $\mu^*(u\ge 0)=0$, then we say that u

is frequently negative. If u is neither frequently positive nor frequently negative, then we say that u is frequently oscillatory.

Lemma 2.1[1] $\mu(\varnothing)=$, $\mu(\Omega)=1$. If A is an arbitrary subset of Ω, then $0\le\mu_*(A)\le\mu^*(A)\le 1$. Particularly, if B is a finite subset of Ω, then $\mu(B)=0$.

Lemma 2.2[1] Let A and B be subsets of Ω, then $\mu^*(A+B)\le\mu^*(A)+\mu^*(B)$. If $A\cap B=\varnothing$, then

$$\mu_*(A)+\mu_*(B)\le\mu_*(A+B)\le\mu_*(A)+\mu^*(B)\le\mu^*(A+B)\le\mu^*(A)+\mu^*(B).$$

Hence, $\mu_*(A)+\mu*(\Omega\setminus A)=1$.

Lemma 2.3[1] Let A and B be subsets of Ω, then

$$\mu^*(A)-\mu^*(B)\le\mu^*(A\setminus B)\le\mu^*(A)-\mu_*(B);$$

$$\mu_*(A)-\mu^*(B)\le\mu_*(A\setminus B)\le\mu_*(A)-\mu_*(B).$$

From Lemma 2.3 we can easily get that

Lemma 2.4[1] Let A and B be subsets of Ω, then

$$\mu_*(A)+\mu^*(B)-\mu^*(A\cdot B)\le\mu^*(A+B)\le\mu^*(A)+\mu^*(B)-\mu_*(A\cdot B);$$

$$\mu_*(A)+\mu^*(B)-\mu^*(A\cdot B)\le\mu_*(A+B)\le\mu_*(A)+\mu^*(B)-\mu_*(A\cdot B).$$

Lemma 2.5[2] For any subset A of Ω, we have

$$\mu^*(X_\alpha^\beta Y_\gamma^\tau A)\le(\beta-\alpha+1)(\tau-\gamma+1)\mu^*(A);$$

$$\mu_*(X_\alpha^\beta Y_\gamma^\tau A)\le(\beta-\alpha+1)(\tau-\gamma+1)\mu_*(A).$$

where α,β,γ and τ are all integers satisfying $\alpha\le\beta,\gamma\le\tau$.

Lemma 2.6[1] Let $A_1,A_2,\cdots,A_n$ be n subsets of Ω, then

$$\mu^*(\sum_{i=1}^n A_i)\le\sum_{i=1}^n\mu^*(A_i)-(n-1)\mu_*(\prod_{i=1}^n A_i),$$

$$\mu_*(\sum_{i=1}^n A_i)\le\mu_*(A_1)+\sum_{i=2}^n\mu^*(A_i)-(n-1)\mu_*(\prod_{i=1}^n A_i).$$

Lemma 2.7[1] Let A and B be subsets of Ω, if $\mu^*(A)+\mu_*(B)>1$, then $A\cdot B$ is an infinite set.

3. Unsaturated Solutions

Let $\{u_{m,n}\}$ be a solution of equation (1.1), if there exists $M \in Z^+$ such that $u_{m,n} > 0$ for any $m, n \geq M$, then we call $\{u_{m,n}\}$ to be an eventually positive solution of equation (1.1). Similarly, we can define eventually negative solution, eventually non-positive solution, eventually non-negative solution of equation (1.1). If the solution of equation (1.1) is neither eventually positive nor eventually negative, then we call that the solution of equation (1.1) is oscillatory. Obviously, the solution of equation (1.1) is oscillatory if and only if for any subset $\{(m,n) \in N^2 : m,n \geq -1\}$ of Ω, there exists (m_1, n_1), (m_2, n_2) such that $u_{m_1,n_1} \cdot u_{m_2,n_2} \leq 0$.

Definition 3.1[4] Assume that $u = \{u_{m,n}\}_{(m,n)\in\Omega}$ is an arbitrary real double sequence, if $\mu^*(u > 0) = \omega \in (0,1)$, then we call that u has unsaturated upper positive part. If $\mu_*(u > 0) = \omega \in (0,1)$, then we call that u has unsaturated lower positive part. If $\mu^*(u > 0) = \mu_*(u > 0) = \omega \in (0,1)$, then we call that u has unsaturated positive part.

Similarly, we can define that u has unsaturated negative part. Obviously, if $u = \{u_{m,n}\}$ is eventually positive or eventually negative, then $\mu^*(u > 0) = 1$ or $\mu^*(u > 0) = 0$. So if the sequence $u = \{u_{m,n}\}$ has unsaturated upper positive part, then it must never be eventually positive or eventually non-positive, that is to say, u is oscillatory.

We then discuss whether the solution of equation (1.1) has unsaturated upper positive part. For any double sequence $\{v_{i,j}\}$ defining on Ω, define a level set $\{(i,j) \in \Omega \mid v_{i,j} > c\}$ as $(v > c)$. Similarly, we can define $(v \geq c)$, $(v < c)$ and $(v \leq c)$.

For convenience, we assume $Z[-1,\infty) \times Z[-1,\infty) = \Omega$. For any real double sequence $\{u_{m,n}\}_{(m,n)\in\Omega}$ define the following two partial differences:

$$\nabla_1 u_{m,n} = u_{m+1,n} - u_{m,n}; \qquad \nabla_2 u_{m,n} = u_{m,n+1} - u_{m,n}$$

Lemma 3.1[6] Let $x, y \geq 0$ and $p, q > 0$, if $\frac{1}{p} + \frac{1}{q} = 1$, then $\frac{1}{p}x + \frac{1}{q}y \geq x^{\frac{1}{p}} y^{\frac{1}{q}}$.

Lemma 3.2 Assume that there exists $m_0 \geq 1$ and $n_0 \geq 2l_1$ such that $p_{m,n} \geq 0$, $q_{m,n} \geq 0$, where $(m,n) \in Z[m_0 - 1, m_0 + 2k_1] \times Z[n_0 - 2l_1, n_0 + 1]$. If $u_{m,n}$ is any solution of equation (1.1), then

$$\nabla_1 u_{m-1,n} \geq 0, \qquad \nabla_2 u_{m,n} \leq 0,$$

for $u_{m,n} \geq 0$, where $(m,n) \in Z[m_0, m_0 + k_1] \times Z[n_0 - l_{1,} n_0]$; and

$$\nabla_1 u_{m-1,n} \leq 0, \qquad \nabla_2 u_{m,n} \geq 0,$$

for $u_{m,n} \leq 0$, where $(m,n) \in Z[m_0, m_0 + k_1] \times Z[n_0 - l_1, n_0]$.

Proof. If $u_{m,n} \geq 0$, $(m,n) \in Z[m_0 - 1, m_0 + 2k_1] \times Z[n_0 - 2l_1, n_0 + 1]$, then from equation (1.1) we have

$$u_{m,n} = u_{m-1,n} + u_{m,n+1} + p_{m,n} u^{\alpha}_{m+k_1,n-l_1} + q_{m,n} u^{\beta}_{m+k_2,n-l_2} \geq u_{m-1,n} + u_{m,n+1}.$$

So

$$\nabla_1 u_{m-1,n} \geq 0, \qquad \nabla_2 u_{m,n} \leq 0,$$

where $(m,n) \in Z[m_0, m_0 + k_1] \times Z[n_0 - l_{1,} n_0]$.

Similarly, if $u_{m,n} \leq 0$, $(m,n) \in Z[m_0 - 1, m_0 + 2k_1] \times Z[n_0 - 2l_1, n_0 + 1]$, we have

$$\nabla_1 u_{m-1n,} \leq 0, \qquad \nabla_2 u_{m,n} \geq 0,$$

where $(m,n) \in Z[m_0, m_0 + k_1] \times Z[n_0 - l_1, n_0]$.

Assume that $\theta = \min\left\{\frac{\beta-\alpha}{\beta-1}, \frac{\beta-\alpha}{1-\alpha}\right\}$, since $\alpha \in [0,1)$, $\beta \in (1,\infty)$, then $\theta > 1$. Set

$$p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}} = \left\{ p_{m,n}^{\frac{\beta-1}{\beta-\alpha}} q_{m,n}^{\frac{1-\alpha}{\beta-\alpha}} \right\}_{m,n\in Z[-1,\infty),}$$

then under the assumption $(*)$, $p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}$ makes sense. If $p_{m,n} \geq 0$, $q_{m,n} \geq 0$, then the condition $(*)$ can be deleted.

Theorem 3.1 Assume that there exists ω_1, ω_2, ω_3 and $\omega \in (0,1)$ such that

$$\mu^*(p<0) = \omega_1, \quad \mu^*(q<0) = \omega_2, \quad \mu_*[(p<0)\cdot(q<0)] = \omega_3,$$

$$\mu_*(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}} > 1) > 4(k_1+1)(l_1+1)(\omega_1 + \omega_2 + \omega - \omega_3).$$

then any solution of equation (1.1) has unsaturated upper positive part.

Proof. Let $u=\{u_{m,n}\}$ be any solution of equation (1.1), then $\mu^*(u>0)\in(\omega,1)$. Otherwise $\mu^*(u>0)\le\omega$ or $\mu^*(u>0)=1$.

If $\mu^*(u>0)\le\omega$, then from Lemma 2.2 to Lemma 2.6 we have

$$1=\mu^*\left\{\Omega\setminus X^1_{-2k_1}Y^{2l_1}_{-1}[(p<0)+(q<0)+(u>0)]\right\}+\mu_*\left\{X^1_{-2k_1}Y^{2l_1}_{-1}[(p<0)+(q<0)+(u>0)]\right\}$$

$$\le\mu^*\left\{\Omega\setminus X^1_{-2k_1}Y^{2l_1}_{-1}[(p<0)+(q<0)+(u>0)]\right\}$$

$$+4(k_1+1)(l_1+1)\left\{\mu_*[(p<0)+(q<0)]+\mu^*(u>0)\right\}$$

$$\le\mu^*\left\{\Omega\setminus X^1_{-2k_1}Y^{2l_1}_{-1}[(p<0)+(q<0)+(u>0)]\right\}$$

$$+4(k_1+1)(l_1+1)\left\{\mu^*(p<0)+\mu^*(q<0)+\mu^*(u>0)-\mu_*[(p<0)\cdot(q<0)]\right\}$$

$$\le\mu^*\left\{\Omega\setminus X^1_{-2k_1}Y^{2l_1}_{-1}[(p<0)+(q<0)+(u>0)]\right\}+4(k_1+1)(l_1+1)(\omega_1+\omega_2+\omega-\omega_3)$$

$$<\mu^*\left\{\Omega\setminus X^1_{-2k_1}Y^{2l_1}_{-1}[(p<0)+(q<0)+(u>0)]\right\}+\mu_*(\theta p^{\frac{\beta-1}{\beta-\alpha}}q^{\frac{1-\alpha}{\beta-\alpha}}>1).$$

then from Lemma 2.7 we know that the intersection

$$\{\Omega\setminus X^1_{-2k_1}Y^{2l_1}_{-1}[(p<0)+(q<0)+(u>0)]\}\cdot(\theta p^{\frac{\beta-1}{\beta-\alpha}}q^{\frac{1-\alpha}{\beta-\alpha}}>1)$$

is an infinite subset of Ω. So according to Lemma 3.2, there exists $m_0\ge1$, $n_0\ge 2l_1$ such that

$$\theta p_{m_0,n_0}^{\frac{\beta-1}{\beta-\alpha}}q_{m_0,n_0}^{\frac{1-\alpha}{\beta-\alpha}}>1, \tag{3.2}$$

$$p_{m,n}\ge0,\quad q_{m,n}\ge0,\quad u_{m,n}\le0,\ (m,n)\in Z[m_0-1,m_0+2k_1]\times Z[n_0-2l_1,n_0+1]. \tag{3.3}$$

From (3.2) and Lemma 3.2 again, we have

$$\nabla_1 u_{m-1,n}\le0,\quad \nabla_2 u_{m,n}\ge0,\quad (m,n)\in Z[m_0,m_0+k_1]\times Z[n_0-l_1,n_0].$$

Therefore $u_{m_0+k_1,n_0-l_1}\ge u_{m_0,n_0-l_2}\ge u_{m_0,n_0}>0.$

It follows from equation (1.1) and Lemma 3.2 that

$$0 = u_{m_0-1,n_0} + u_{m_0,n_0+1} - u_{m_0,n_0} + p_{m_0,n_0} \left|u_{m_0+k_1,n_0-l_1}\right|^{\alpha} \operatorname{sgn} u_{m_0+k_1,n_0-l_1}$$

$$+ q_{m_0,n_0} \left|u_{m_0+k_2,n_0-l_2}\right|^{\beta} \operatorname{sgn} u_{m_0+k_2,n_0-l_2}$$

$$\le u_{m_0-1,n_0} + u_{m_0,n_0+1} - u_{m_0,n_0} - \left(p_{m_0,n_0} \left|u_{m_0+k_2,n_0-l_2}\right|^{\alpha} + q_{m_0,n_0} \left|u_{m_0+k_2,n_0-l_2}\right|^{\beta} \right)$$

$$\le u_{m_0-1,n_0} + u_{m_0,n_0+1} - u_{m_0,n_0} - \theta p_{m_0,n_0}^{\frac{\beta-1}{\beta-\alpha}} q_{m_0,n_0}^{\frac{1-\alpha}{\beta-\alpha}} \left|u_{m_0+k_2,n_0-l_2}\right|$$

$$\le u_{m_0-1,n_0} + u_{m_0,n_0+1} - u_{m_0,n_0} + \theta p_{m_0,n_0}^{\frac{\beta-1}{\beta-\alpha}} q_{m_0,n_0}^{\frac{1-\alpha}{\beta-\alpha}} u_{m_0,n_0}$$

$$\le -u_{m_0,n_0} + \theta p_{m_0,n_0}^{\frac{\beta-1}{\beta-\alpha}} q_{m_0,n_0}^{\frac{1-\alpha}{\beta-\alpha}} u_{m_0,n_0}$$

$$= [-1 + \theta p_{m_0,n_0}^{\frac{\beta-1}{\beta-\alpha}} q_{m_0,n_0}^{\frac{1-\alpha}{\beta-\alpha}}] u_{m_0,n_0}.$$

Since $u_{m_0,n_0} \le 0$, then $\theta p_{m_0,n_0}^{\frac{\beta-1}{\beta-\alpha}} q_{m_0,n_0}^{\frac{1-\alpha}{\beta-\alpha}} \le 1$, which contradicts (3.2).

Next, assume $\mu^*(u > 0) = 1$, according to Lemma 2.2 we have $\mu_*(u \le 0) = 0$. From Lemma 2.2 to Lemma 2.6, we have

$$1 = \mu^* \left\{ \Omega \setminus X_{-2k_1}^{1} Y_{-1}^{2l_1} [(p<0)+(q<0)+(u \le 0)] \right\} + \mu_* \left\{ X_{-2k_1}^{1} Y_{-1}^{2l_1} [(p<0)+(q<0)+(u \le 0)] \right\}$$

$$\le \mu^* \left\{ \Omega \setminus X_{-2k_1}^{1} Y_{-1}^{2l_1} [(p<0)+(q<0)+(u \le 0)] \right\}$$

$$+ 4(k_1+1)(l_1+1) \mu_* [(p<0)+(q<0)+(u \le 0)]$$

$$\le \mu^* \left\{ \Omega \setminus X_{-2k_1}^{1} Y_{-1}^{2l_1} [(p<0)+(q<0)+(u \le 0)] \right\}$$

$$+ 4(k_1+1)(l_1+1) \left\{ \mu^*(p<0) + \mu^*(q<0) + \mu_*(u \le 0) - \mu_*[(p<0)\cdot(q<0)] \right\}$$

$$\le \mu^* \left\{ \Omega \setminus X_{-2k_1}^{1} Y_{-1}^{2l_1} [(p<0)+(q<0)+(u \le 0)] \right\} + 4(k_1+1)(l_1+1)(\omega_1 + \omega_2 - \omega_3)$$

$$< \mu^* \left\{ \Omega \setminus X_{-2k_1}^{1} Y_{-1}^{2l_1} [(p<0)+(q<0)+(u \le 0)] \right\} + \mu_*(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}} > 1).$$

So according to Lemma 2.7 we know that the intersection

$$\{\Omega \setminus X_{-2k_1}^{1} Y_{-1}^{2l_1} [(p<0)+(q<0)+(u \le 0)]\} \cdot (\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}} > 1)$$

is an infinite subset of Ω. Similar to the above discussion, $\mu^*(u > 0) = 1$ does not hold, so the conclusion is correct.

Theorem 3.2 Assume that there exist constants $\omega_1,\ \omega_2, \omega_3$ and $\omega \in (0,1)$, such that

$$\mu^*(p<0)=\omega_1,\quad \mu^*(q<0)=\omega_2,\quad \mu_*(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}>1)=\omega_3,$$

$$\mu_*[(p<0)\cdot(q<0)\cdot(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)] > \frac{\omega_1+\omega_2+\omega+\omega_3}{2}-\frac{1}{8(k_1+1)(l_1+1)},$$

then any solution of equation (1.1) has unsaturated upper positive part.

Proof. To prove $\mu^*(u>0)\in(\omega,1)$. Similar to the proof of Theorem 3.1 , we only need to prove $\mu^*(u>0)\le\omega$ and $\mu^*(u>0)=1$.

Firstly, assume $\mu^*(u>0)\le\omega$, then form Lemma 2.2 to Lemma 2.6 we have

$$\mu^*\{\Omega\setminus X_{-2k_1}^{1}Y_{-1}^{2l_1}[(p<0)+(q<0)+(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)+(u>0)]\}$$

$$\begin{aligned}
&=1-\mu_*\{X_{-2k_1}^{1}Y_{-1}^{2l_1}[(p<0)+(q<0)+(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)+(u>0)]\}\\
&\ge 1-4(k_1+1)(l_1+1)\{\mu_*[(p<0)+(q<0)+(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)]+\mu^*(u>0)\}\\
&\le 1-4(k_1+1)(l_1+1)\{\mu^*(p<0)+\mu^*(q<0)+\mu_*(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)+\mu^*(u>0)\\
&\quad -2\mu_*[(p<0)\cdot(q<0)\cdot(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)]\}\\
&>0
\end{aligned}$$

From Lemma 2.7 we know that $\Omega\setminus X_{-2k_1}^{1}Y_{-1}^{2l_1}[(p<0)+(q<0)+(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)+(u>0)]$ is an infinite set. So according to Lemma 3.2, there exist $m_0\ge 1,\ n_0\ge 2l_1$ such that

$$\theta p_{m_0,n_0}^{\frac{\beta-1}{\beta-\alpha}} q_{m_0,n_0}^{\frac{1-\alpha}{\beta-\alpha}}>1,$$

$$p_{m,n}\ge 0,\ q_{m,n}\ge 0,\ u_{m,n}\le 0,\ (m,n)\in Z[m_0-1,m_0+2k_1]\times Z[n_0-2l_1,n_0+1].$$

Similar to Theorem 3.1, we get the contradiction. So $\mu^*(u>0)\le\omega$ does not hold, then $\mu^*(u>0)\ne 1$, i.e., $\mu^*(u>0)\in(\omega,1)$. Hence the theorem is proved.

Theorem 3.3 Assume that there exist constants $\omega_1,\ \omega_2, \omega_3, \omega_4$ and $\omega\in(0,1)$ such that

$$\mu^*(p<0)=\omega_1,\quad \mu^*(q<0)=\omega_2,\quad \mu^*(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)=\omega_3,$$

$$\mu_*[(p<0)\cdot(q<0)\cdot(\theta p^{\frac{\beta-1}{\beta-\alpha}} q^{\frac{1-\alpha}{\beta-\alpha}}\le 1)]=\omega_4$$

Satisfying $4(k_1+1)(l_1+1)(\omega_1+\omega_2+\omega+\omega_3-2\omega_4)<1$, then any solution of equation (1.1) has unsaturated upper positive part.

Proof. Similar to the proof of Theorem 3.2.

Example. In equation (1.1), set $\alpha=\frac{1}{3}$, $\beta=\frac{4}{3}$, $p_{m,n}=3^n$, $q_{m,n}=1$, $k_1=3$, $k_2=2$, $l_1=2$, $l_2=1$, $\theta=\min\{\frac{\beta-\alpha}{\beta-1},\frac{\beta-\alpha}{1-\alpha}\}$, then the equation should be

$$u_{m,n}=u_{m+1,n}+u_{m,n+1}+3^n\left|u_{m+3,n-2}\right|^{\alpha}\operatorname{sgn}u_{m+3,n-2}+\left|u_{m+2,n-1}\right|^{\beta}\operatorname{sgn}u_{m+2,n-1}.$$

Obviously,

$$\mu_*(\theta p^{\frac{\beta-1}{\beta-\alpha}}q^{\frac{1-\alpha}{\beta-\alpha}}>1)=1,\qquad \mu^*(p<0)=\mu^*(q<0)$$

$$\mu_*((p<0)\cdot(q<0))=\mu^*(\theta p^{\frac{\beta-1}{\beta-\alpha}}q^{\frac{1-\alpha}{\beta-\alpha}}>1)=1.$$

Moreover, $\mu_*((p<0)\cdot(q<0)\cdot(\theta p^{\frac{\beta-1}{\beta-\alpha}}q^{\frac{1-\alpha}{\beta-\alpha}}\le 1))=0.$

Then from Theorems 3.2 and 3.3, any solution of equation (1.1) has unsaturated upper positive part.

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