COMMON FIXED POINT THEOREM IN b_2-METRIC SPACES

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ABSTRACT

We establish a unique common fixed point theorem for two pair of weekly compatible maps satisfying a contractive condition in a complete b_2-metric space. When the following have been proved, I recommend it to be published, which extends and generalizes some known results in metric space to b_2-metric space.

Keywords: Common fixed point; complete b_2-metric space; weekly compatible maps.

1 Introduction

Fixed point theory has been studied by many authors for its useful function in a variety of areas. In 1992, a polish mathematician, Banach, proved a theorem known as Banach contraction principle [1]. This principle presents useful results in nonlinear analysis, functional analysis and topology. The concept of weakly commuting has been introduced by Sesssa S [2]. Years later, Gerald Jungck [3] introduced weakly compatible mappings ,which are more generalized commuting mappings.

In this paper, we present fixed point results for two pair of mappings satisfying a contractive type condition by using the concept of weakly compatible mappings in a complete generalized metric space, which is called b_2-metric space [5] and this space was generalized from both 2-metric space [6-8] and b-metric space [9-10].

2 Preliminaries

The following definitions will be needed to present before giving our results.
Definition 2.1 [2] Let f and g be two self-maps on a set X. Maps f and g are said to be commuting if $fgx = gfx$ for all $x \in X$.
Definition 2.2 [4] Let f and g be two self-maps on a set X. If $fx = gx$, for some x of X, then x is called coincidence point of f and g.
Definition 2.3 [4] Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at coincidence points. That is , if $fx = gx$ for some $x \in X$, then $fgx = gfx$.
Lemma 2.4 [4] Let f and g be weakly compatible self mappings of a set X. If f and g have a unique point of coincidence, that is, $\omega = fx = gx$, then ω is the unique common fixed point of f and g.
Definition 2.5 [5] Let X be a nonempty set, $s \geq 1$ be a real number and let $d : X \times X \times X \to R$ be a map satisfying the following conditions:
1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$.

2. If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$,
3. The symmetry:
$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, x, y)$ for all $x, y, z \in X$.
1. The rectangle inequality:
$d(x, y, z) \le s[d(x, y, a) + d(y, z, a) + d(z, x, a)]$, for all $x, y, z, a \in X$.

Then d is called a b_2 metric on X and (X, d) is called a b_2 metric space with parameter s. Obviously, for $s = 1$, b_2 metric reduces to 2-metric.

Definition 2.6 [5] Let $\{x_n\}$ be a sequence in a b_2 metric space (X, d).
(1). A sequence $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if all $a \in X$ $\lim_{n\to\infty} d(x_n, x, a) = 0$.
(2). $\{x_n\}$ is Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$, when $n, m \to \infty$. for all $a \in X$.
(3). (X, d) is said to be -complete if every b_2-Cauchy sequence is a b_2-convergent sequence.

Definition 2.7 [5] Let (X, d) and (X', d') be two b_2-metric spaces and let $f : X \to X'$ be a mapping. Then f is said to be b_2-continuous,at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z, x, a) < \delta$ for all $a \in X$ imply that $d'(fz, fx, a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.

Definition 2.8 [5] Let (X, d) and (X', d') be two b_2-metric spaces. Then a mapping $f : X \to X'$ is b_2-continuous at a point $x \in X'$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-convergent to x, $\{fx_n\}$ is b_2-convergent to $f(x)$.

Definition 2.9 [6-8] Let X be an nonempty set and let $d : X \times X \times X \to R$ be a map satisfying the following conditions:
1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \ne 0$.
2. If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$,
3. The symmetry:
$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, x, y)$ for all $x, y, z \in X$.
4. The rectangle inequality:
$d(x, y, z) \le d(x, y, a) + d(y, z, a) + d(z, x, a)$ for all $x, y, z, a \in X$.

Then d is called a 2 metric on X and (X, d) is called a 2 metric space.

Definition 2.10 [9-10] Let X be a nonempty set and $s \ge 1$ be a given real number. A function $d : X \times X \to R^+$ is a b metric on X if for all $x, y, z \in X$, the following conditions hold:
2. $d(x, y) = 0$ if and only if $x = y$.

3. $d(x,y)=d(y,x)$.
4. $d(x,y)\le s[d(x,y)+d(y,z)]$.

In this case, the pair (X,d) is called a b metric space.

3 Main results

Theorem 3.1. Let (X,d) be a complete b_2-metric space, and $P,Q,S,T:X\to X$ are four mappings, satisfying the following conditions:

(a). $T(X)\subseteq P(X)$ and $S(X)\subseteq Q(X)$; Both P and Q are surjections.

(b). $d(Sx,Ty,a)\le c(\lambda(x,y,a))$.

Where $\lambda(x,y,a)=\max\{d(Px,Qx,a),d(Px,Sx,a),d(Qx,Ty,a)\}$ for all $x,y\in X$ and $0\le c<\frac{1}{s}$.

(c) $.(S,P)$ and (T,Q) are weakly compatible.

Then S,P,Q and T have a unique common fixed point in X.

Proof In this part, we will show that $\lim\limits_{n\to} d(y_{n+1},y_n,a)=0$.

Let x_0 be an arbitrary point in X and construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_n=Qx_{n+1}=Sx_n,$$
$$y_{n+1}==Px_{n+2}=Tx_{n+1}.$$

From (b), we have

$$d(y_n,y_{n+1},a)=d(Sx_n,Tx_{n+1},a)\le c\lambda(x_n,x_{n+1},a). \qquad (3.1)$$

where

$$\begin{aligned}&\lambda(x_n,x_{n+1},a)\\&=\max\{d(Px_n,Qx_{n+1},a),d(Px_n,Sx_n,a),d(Qx_{n+1},Tx_{n+1},a)\}\\&=\max\{d(Tx_{n-1},Sx_n,a),d(Tx_{n-1},Sx_n,a),d(Sx_n,Tx_{n+1},a)\}\\&=\max\{d(Tx_{n-1},Sx_n,a),d(Sx_n,Tx_{n+1},a)\}\\&=\max\{d(y_{n-1},y_n,a),d(y_n,y_{n+1},a)\}.\end{aligned}$$

Assume $\lambda(x_n,x_{n+1},a)=d(y_n,y_{n+1},a)$ and from (3.1) we have ,

$$d(y_n,y_{n+1},a)<cd(y_n,y_{n+1},a),$$

which is impossible. Then we get $\lambda(x_n,x_{n+1},a)=d(y_{n-1},y_n,a)$ also from (3.1) we get

$$d(y_n,y_{n+1},a)<cd(y_{n-1},y_n,a) \quad . \qquad (3.2)$$

This implies that the sequence $\{d(y_n,y_{n+1},a)\}$ is decreasing and it must converge to $r\ge 0$. Therefore as $n\to\infty$, from (3.2) we get $r\le cr$, this gives us that $r=0$, then the result is obtained:

$$\lim_{n\to\infty} d(y_{n+1},y_n,a)=0 \quad . \qquad (3.3)$$

Then we show that $d(y_i,y_j,y_k)=0$.

From part 2 of Definition 2.5, we have $d(x_m,x_m,x_{m-1})=0$. Since $\{d(x_n,x_{n+1},a)\}$ is decreasing, we get $d(x_n,x_{n+1},a)=0$ from the assumption that $d(x_{n-1},x_n,a)=0$, then it is easy to get

$$d(x_n,x_{n+1},x_m)=0 \quad , \qquad \text{for all } n+1\ge m. \qquad (3.4)$$

ISSN 2059-9951

For $0 \le n+1 < m$, we get $m-1 \ge n+1$ and that is $m-2 \ge n$, from (3.4)

$$d(x_{m-1}, x_m, x_{n+1}) = d(x_{m-1}, x_m, x_n) = 0, \quad (3.5)$$

For (3.5) and triangular inequality, we have

$$\begin{aligned} d(x_n, x_{n+1}, x_m) &\le sd(x_n, x_{n+1}, x_{m-1}) + sd(x_{n+1}, x_m, x_{m-1}) \\ &\quad + d(x_m, x_n, x_{m-1}) \\ &= sd(x_n, x_{n+1}, x_{m-1}). \end{aligned}$$

And since $d(x_n, x_{n+1}, x_{n+1}) = 0$, and from the inequality above,

$$d(x_{n+1}, x_n, x_m) \le s^{m-n-1} d(x_{n+1}, x_{n+1}, x_n) = 0, \text{ for all } 0 \le n+1 \le m. \quad (3.6)$$

Now for all $i, j, k \in N$, now we consider the condition of $j > i$, from the above equation

$$d(x_{j-1}, x_j, x_i) = d(x_k, x_{j-1}, x_j) = 0. \quad (3.7)$$

From (3.7) and triangular inequality, therefore

$$\begin{aligned} d(x_i, x_k, x_j) &\le s[d(x_i, x_j, x_{j-1}) + d(x_j, x_{k-1}, x_k) + d(x_i, x_{j-1}, x_k)] \\ &\le \Lambda \\ &\le s^{j-1} d(x_i, x_k, x_i) \\ &= 0. \end{aligned}$$

In conclusion, the result below is gotten

$$d(x_j, x_k, x_i) = 0, \text{ for all } i, j, k \in N. \quad (3.8)$$

Now we prove that $\{y_n\}$ is a Cauchy sequence.

Suppose to the contrary, that is, $\{y_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{n_i\}$ and $\{m_i\}$ such that $i < m_i < n_i$ and

$$d(y_{m_i}, y_{n_i}, a) \ge \varepsilon \text{ and } d(y_{m_i}, y_{n_i-1}, a) < \varepsilon. \quad (3.9)$$

From the part 4 of Definition 2.5 and (3.8), we get

$$\begin{aligned} d(y_{m_i}, y_{n_i}, a) &\le s[d(y_{m_i}, y_{m_i+1}, a) + d(y_{m_i+1}, y_{n_i}, a) + d(y_{m_i}, y_{n_i}, y_{m_i+1})] \\ &\le s[d(y_{m_i}, y_{m_i+1}, a) + d(y_{m_i+1}, y_{n_i}, a)]. \end{aligned}$$

Taking $i \to \infty$, from (3.3) and (3.9) we have

$$\frac{\varepsilon}{s} \le \lim_{n\to\infty} d(y_{m_i+1}, y_{n_i}, a). \quad (3.10)$$

From (b), we get

$$d(y_{n_i}, y_{m_i+1}, a) = d(Sx_{n_i}, Tx_{m_i+1}, a) \le c\lambda(x_{n_i}, y_{m_i+1}, a). \quad (3.11)$$

Since

$$\begin{aligned} \lim_{n\to\infty} \lambda(x_{n_i}, x_{m_i+1}, a) &= \max\{\lim_{n\to\infty} d(Px_{n_i}, Qx_{m_i+1}, a), \lim_{n\to\infty} d(Px_{n_i}, Sx_{m_i+1}, a), \\ &\quad \lim_{n\to\infty} d(Qx_{m_i+1}, Tx_{m_i+1}, a) \\ &= \max\{\lim_{n\to\infty} d(y_{n_i-1}, y_{m_i}, a), \lim_{n\to\infty} d(y_{n_i-1}, y_{n_i}, a), \lim_{n\to\infty} d(y_{m_i+1}, y_{m_i}, a)\} \\ &= \lim_{n\to\infty} d(y_{n_i-1}, y_{m_i}, a). \end{aligned}$$

And by (3.11) we have

$$\lim_{n\to\infty} d(y_{n_i}, y_{m_i+1}, a) \le \lim_{n\to\infty} c(d(y_{n_i-1}, y_{m_i}, a)). \quad (3.12)$$

Again taking $i \to \infty$ by (3.9) and (3.12) we get

$$\frac{\varepsilon}{s} \le \lim_{n\to\infty} d(y_{m_i+1}, y_{n_i}, a) \le c\varepsilon < \frac{\varepsilon}{s}. \quad (3.13)$$

Which is a contraction. Therefore $\{y_n\}$ is a Cauchy sequence in X.
Since X is complete, there exists a point $z \in X$ such that $n \to \infty$, $\{y_n\} \to z$.
Thus $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qx_{n+1} = z$ and $\lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Px_{n+2} = z$.
That is $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Qx_{n+1} = \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Px_{n+2} = z$. From $T(X) \subseteq P(X)$ and P is a surjection, there exists a point u in X such that $z = Pu$, then from (b), we get

$$\begin{aligned} d(Su,z,a) &\le s[d(Su,Tx_{n+1},a) + d(Tx_{n+1},z,a) + d(Tx_{n+1},Su,z)] \\ &\le s[c\lambda(u,x_{n+1},a) + d(Tx_{n+1},z,a) + d(Tx_{n+1},Su,a)], \end{aligned}$$

where

$$\begin{aligned} \lambda(u,x_{n+1},a) &= \max\{d(Pu,Qx_{n+1},a), d(Pu,Su,a), d(Qx_{n+1},Tx_{n+1},a)\} \\ &= \max\{d(z,Sx_n,a), d(z,Su,a), d(Sx_n,Tx_{n+1},a)\}. \end{aligned}$$

We take $n \to \infty$, we get

$$\lambda(u,x_{n+1},a) = \max\{d(z,z,a), d(z,Su,a), d(z,z,a)\} = d(z,Su,a).$$

Therefore as $n \to \infty$, $d(Su,z,a) \le sc(d(z,Su,a))$.

Assume there exists $a \in X$ such that $d(Su,z,a) > 0$ then we get $\frac{1}{s} \le c$ from the above inequality, which is contraction with $c < \frac{1}{s}$. Thus $Su = z$, furthermore $Pu = Su = z$. So P and S have a coincidence point u in X. Since P and S are weakly compatible, $SPu = PSu$ that is $Sz = Pz$.

From $S(X) \subseteq Q(X)$ and Q is a surjection, there exists a point v in X such that $z = Qv$, then from (b), we get

$$d(Tv,z,a) \le c\lambda(u,v,a),$$

where

$$\begin{aligned} \lambda(u,v,a) &= \max\{d(Pu,Qv,a), d(Pu,Su,a), d(Qv,Tv,a)\} \\ &= \max\{d(z,z,a), d(z,z,a), d(z,Tv,a)\} \\ &= d(z,Tv,a). \end{aligned}$$

Then

$$d(z,Tv,a) \le cd(z,Tv,a).$$

Assume $d(z,Tv,a) > 0$, then we have $1 \le c$,which is contraction with $c < \frac{1}{s} < 1$. Therefore $Tv = Qv = z$. So Q and T have a coincidence point v in X. Since Q and T are weakly compatible, $QTv = TQv$ that is $Qz = Tz$.

Now we prove that z is a fixed point of S. By (b), we get

$$d(Sz,z,a) = d(Sz,Tv,a) \le c\lambda(z,v,a),$$

where

$$\begin{aligned} \lambda(z,v,a) &= \max\{d(Pz,Qv,a), d(Pz,Sz,a), d(Qv,Tv,a)\} \\ &= \max\{d(Sz,z,a), d(Sz,Sz,a), d(z,z,a)\} \\ &= d(Sz,z,a). \end{aligned}$$

then we get

$$d(Sz,z,a) \le cd(Sz,z,a).$$

Assume $d(z,Tv,a) > 0$, we get $1 \le c$, which is a contraction. Thus $Sz = Pz = z$.

Now we prove that z is a fixed point of T. Then from (b), we get

$$d(Tz,z,a)=d(Sz,Tz,a)\le c\lambda(z,z,a),$$

where

$$\lambda(z,z,a)=\max\{d(Pz,Qz,a),d(Pz,Sz,a),d(Qz,Tz,a)\}$$
$$=\max\{d(Tz,z,a),d(Tz,Tz,a),d(z,z,a)\}$$
$$=d(Tz,z,a).$$

then we get

$$d(z,Tz,a)\le cd(z,Tv,a).$$

Assume $d(z,Tz,a)>0$, we have $1\le c$, which is a contraction. Thus $Tz=Qz=z$.
So we get z is a common fixed point of P,Q,S,T. From (b), we get

$$d(z,\omega,a)=d(Sz,T\omega,a)\le c\lambda d(z,\omega,a),$$

where

$$\lambda(z,\omega,a)=\max\{d(Pz,Q\omega,a),d(Pz,Sz,a),d(Q\omega,T\omega,a)\}$$
$$=\max\{d(z,\omega,a),d(z,z,a),d(\omega,\omega,a)\}$$
$$=d(z,\omega,a).$$

thus $d(z,\omega,a)\le c\lambda d(z,\omega,a)$.
Suppose that $d(z,\omega,a)>0$, we get $1\le c$, which is a contraction. Thus $z=\omega$, then P,Q,S,T have a unique common fixed point $z\in X$. □

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