ELEMENTARY AND COMPOUND EVENTS PROBABILITY

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ABSTRACT

The objective of this paper is to review some known elementary and compound events. We will review the sampling with and without replacement cases in order to see the effects on the probability. This will also change the weighting process in survey sampling. Finally, we will discuss two useful compound events, the Banach Matchbox Problem and the Poisson Negative Binomial Distribution in compound Events. These two distributions have been found useful in risk analysis. All theories are accompanied with examples to explain the meaning of these theories.

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Banach Matchbox Problem, Dependent and Independent Events,
Elementary and Compound Events, Empty Box, Find Empty Box,
Poisson Negative Binomial Distribution, Probability, With or Without Replacement.

1.INTRODUCTION

An event A is said to be a compound event, if it can be represented as the sum of two events that are both different from A: $A = B + C, B \neq A, C \neq A.$ Events that do not permit any such representation are said to be elementary events. There are some interesting relationships between elementary and compound events. We provide a short list, but no proven of these facts: 1) It can be shown that the product of two distinct elementary events is 0; 2) We know that for every compound event B, there exists an elementary event A such that $A \subset B$ holds; 3) In an algebra consisting of a finite number of events,every event can be represented as a sum of elementary events. This representation is unique except for the order of the events. 4) The number of events of a finite algebra of events is necessarily a power of 2. Instead of proving these facts we will discuss some of generally useful events that may relate to other areas study. The events with or without replacement could change the probability, and thus effect the weighing process in a sample survey. In section 3, we give three classical examples that have been found useful in risk analysis. In general, it is not easy to find their probabilities. In example 3.1 and 3.2, we study the Banach Matchbox Problem and more complex situation, first empty but not the first one found empty. These situations can be extended to real life applications. In example 3.3, we had applied the Poisson distribution compound with Negative Binomial Distribution. The reason for using this distribution is proved mathematically, and not assumed. We give this proven fact in our concluding remarks.

2. Elementary Events

Draw an ordered sample of size k, $0 \leq k \leq n$, without replacement from a population of size n. The total number of possible samples denoted by $n_{(k)}$, is

$$n_{(k)} = n(n-1)(n-2).....(n-k+1)$$

$n_{(k)}$ is also known as the permutation of n objects taken k

at a time. By a permutation of n objects we mean any arrangement of these n objects. This implies that the total number of permutations of n objects is $n_{(n)} = n!$. However, if

we concern the sampling with replacement, then the total number of samples of size k with replacement from a population of size n is n^k. We apply these methods to the following examples.

Example 2.1 An elevator starts with 10 passengers and stops at 15 floors. Find the probability that no two passengers leave at the same floor.

Let A: event that no two passengers get off on the same floor. Then the number of favorable event A is

$$15*14*13*12*11*10*9*8*7*6*$$

and sample space number $\Omega = 15^{10}$. Therefore the probability of event A is

$$P(A) = \frac{\#(A)}{\#(\Omega)} = \frac{15*14*13*12*11*10*9*8*7*6}{15^{10}} = 0.0189$$

Example 2.2 Suppose in a population of r elements, a random sample of size n is taken. Find the probability that none of k prescribed elements is in the sample if the method used is :(a) sampling without replacement; (b) sampling with replacement.

a. Given that the population size is r, the sample size is n, and the sampling is done without replacement. So the number of sample space is

$$\#(\Omega) = r_{(n)} = r(r-1).....(r-n+1)$$

Define A: the event that none of k prescribed objects is in the sample. If the k prescribed objects are excluded from the sample, then the sample of n objects must be selected from the remaining (r-k) objects. This can be done $(r-k)_{(n)}$ ways. Thus, the number of favored events $A = (r-k)_{(n)}$ and

$$P(A) = \frac{(r-k)_{(n)}}{r_{(n)}}.$$

b. In this case, the sampling is done with replacement and the number of sample space $\#(\Omega) = r^n$. Define A event the same as part (a). We have the number of favored event A: $\#(A) = (r-k)^n$. So that $P(A) = \frac{(r-k)^n}{r^n} = (1-\frac{k}{n})^n$.

Example 2.3: Suppose n balls are distributed into n boxes so that all of the n^n possible arrangements are equally likely. Compute the probability that only box 1 is empty.

The probability space in this case consists of n^n equally likely points. Let A be the event that only box 1 is empty. This can happen only if the n balls are in the remaining

n-1 boxes in such a manner that no box is empty. Thus, exactly one of these (n-1) boxes must have two balls, and the remaining (n-2) boxes must have exactly one ball each. Let B_j be the event that box j, j=2,3,…n has two balls, box 1 has no balls, and the remaining (n-2) boxes have exactly one ball each. Then the B_j are disjoint events and $A = \bigcup_{j=2}^{n} B_j$. To compute $P(B_j)$ observe that the two balls put in box j can be chosen from n balls in $\binom{n}{2}$ ways. The (n-2) balls in the remaining (n-2) boxes can be rearranged in (n-2) ways. Thus the number of distinct ways that we can put two balls into box j, no ball in box 1, and exactly one ball in each of the remaining boxes is $\binom{n}{2}(n-2)!$ so $P(B_j) = \dfrac{\binom{n}{2}(n-2)!}{n^n}$

and consequently

$$P(A) = \sum_{j=2}^{n} P(B_j) = \frac{(n-1)\binom{n}{2}(n-2)!}{n^n} = \frac{\binom{n}{2}(n-1)!}{n^n}$$

n	5	10	15	20
P(A)	*0.0768*	*0.0016*	$0.2x10^{-4}$	$0.22x10^{-6}$

Based on above tabulation, we can see that as the sample size increase from 5 to 20 and the chance of event A occur quickly drop to zero.

3. Define the Random Variables

Instead of defining the events, we sometime prefer to define the random variables in a sample space. In this section we introduce the discrete random variable, namely the negative binomial distribution. In a repeated independent Bernoulli trial with probability p for success until obtaining the r^{th} success, we define the random variable x : number of the trials until the r^{th} success occurs. Suppose the r^{th} success occur at the $(k+r)^{th}$ trial, k=0,1,2,…. This means that there are k F's until the r^{th} success. There are exactly k F's in (k+r-1) trials follows by a success at the $(k+r)^{th}$ trials, where these events occur with probability $\binom{k+r-1}{k} q^k p^{r-1} p$ therefore $P(x = k+r) = \binom{k+r-1}{k} p^r q^k$

$$P(r,k,p) = \binom{k+r-1}{k} p^r q^k \qquad k = 0,1,2,3......, r = 1,2,3,....$$

$$= 0 \qquad otherwise$$

The above Negative Binomial Distribution is also known as the Pascal Distribution. We found it very helpful in finding some difficult probabilities. The next two examples will demonstrate this.

Example 3.1 (Banach's Matchbox Problem)

A mathematician carries two matchboxes, one in each of the two pockets of his coat. Whenever he wishes to light a cigarette, he chooses a pocket at random and uses a match from the box in that pocket. Suppose each box contains N matches to begin with. Find the probability that there are k matches in the other pocket when he finds that the box in one pocket is empty. (Remember to differentiate between empty box is different from finds empty box) Define A : the event that one of the matchboxes is found empty when the other box contains exactly k matches. Let $A_R(A_L):$ the event that the right (left) pocket is found to contain the empty match box while the left (right) pocket contains a matchbox with exactly k matches. Then

$A = A_R \cup A_L$ and since $P(A_R) = P(A_L),\ so\ P(A) = 2P(A_R)$, so let us compute $P(A_R)$. We may also assume the chance to select the right pocket or left pocket is the same. Then p=0.5 for success. Now, finding, for the first time, the box in the right pocket empty means a (N+1) chance for success. Exactly k matches left in the left pocket means that the left pocket has been selected (N-k)times. So we have had (N-k) failures. Thus, if f(r,k,p) denotes the negative binomial density, we have $P(A_R) = f(N+1, N-k, \frac{1}{2})$ and

$$P(A) = 2f(N+1, N-k, \frac{1}{2}) = 2\binom{2N-k}{N-k}(\frac{1}{2})^{N+1}(\frac{1}{2})^{N-k}$$

To show some sample example of computing P(A), we select some N, k and calculate their probabilities follows:

N	10	10	15	15	20	20
k	5	8	7	11	12	17
P(A)	0.0916	0.0161	0.0584	0.0074	0.0116	0.0002

Example 3.2 Find the probability that, at the moment where the first box is emptied and is not found empty, the other contains exactly k matches where k=1,2,….N. Using this result, find the probability x that the box first emptied is not the one first found empty. Define all events the same as the example 3.1, except by changing "is found empty" to "is empty". Similarly, drop the word "found" in defining $"A_R(A_L)"$. Now, finding the box in the right pocket empty means N success, and exactly k matches left in the left pocket means that the left pocket has been selected (N-k)times. So we have had (N-k) failures. Thus, we have $P(A_R) = f(N, N-k, \frac{1}{2})$ and

$$P(A) = 2f(N, N-k, \frac{1}{2}) = 2\binom{2N-k-1}{N-k}(\frac{1}{2})^{N}(\frac{1}{2})^{N-k} = \binom{2N-k-1}{N-1}2^{-2N+k+1}$$

To find the probability x that the box first emptied is not the one first found empty:

$$x = 2\sum_{k=1}^{N}\binom{2N-k-1}{N-1}(\frac{1}{2})^{N}(\frac{1}{2})^{N+1} = 2^{-2N}\sum_{k=1}^{N}\binom{2N-k-1}{N-1}$$

N	5	10
$\sum_{k=1}^{N}\binom{2N-k-1}{N-1}$	126	92378
X	0.1230	0.0881

Finally, we would like to provide another useful example of compound distribution, i.e. Poisson distribution compound with negative binomial distribution.

Example 3.3 During its flight period, the instrument compartment of a spacecraft is reached by r elementary particles with the probability density function, $f(r,\lambda)=\frac{\lambda^r e^{-\lambda}}{r!}$. The conditional probability for each particle to hit a pre-assigned unit equals p. Find the probability that this unit will be hit by (a) exactly k particles; (b)
at least one particle.
Let y random variable the number to hit pre-assigned unit with probability p.

$$P(y=k)=\sum_{r=0}^{\infty}P(N=r,y=k)=\sum_{r=0}^{\infty}P(N=r)P(y=k/N=r)$$

$$=\sum_{r=0}^{\infty}\frac{e^{-\lambda}\lambda^r}{r!}\binom{r-k+k-1}{r-k}p^k(1-p)^{r-k}$$

$$=\sum_{r=0}^{\infty}\frac{e^{-\lambda}\lambda^r}{r!}\frac{(r-1)!}{(r-k)!(k-1)!}(\frac{p}{1-p})^k(1-p)^r$$

$$=\sum_{r=1}^{\infty}\frac{e^{-\lambda}(\lambda(1-p))^r}{(r-k)!(k-1)!}(\frac{p}{1-p})^k$$

$$P(k=0)=\sum_{r=1}^{\infty}\frac{e^{-\lambda}(\lambda(1-p))^r}{r!(-1)!}=e^{-\lambda}e^{\lambda(1-p)}=e^{-\lambda p}$$

$$P(k\geq 1)=1-P(0)=1-e^{-\lambda p}$$

There is a trivial reason that we are not interested in the case when r=0. We use the property of gamma function, $\Gamma(n+1)=n!\ and\ \Gamma(0)=1$, we can calculate -1!=1. In this way we derive that the probability no particle hit the
pre-assigned point is $e^{-\lambda p}$ and at least one particle is $1-e^{-\lambda p}$.

CONCLUDING REMARKS

The experiments result in counting the number of times particular events occur in given times or on given physical objects. For example, we could count the number of phone calls arriving at a switchboard between 1 and 2 p.m., the number of customers that arrive at a ticket window between 12 noon and 1 p.m., or the number of patients arriving a hospital in a certain day. Each count can be looked upon as a random variable associated with an approximate Poisson Process with parameter $\mu>0$, provided the following four conditions are satisfied.

1. Random events in non-overlapping intervals are independent.
2. In an infinitesimal interval of length Δt, the probability of occurrence exactly 1 event is $\mu\Delta t+o(\Delta t)$
3. In the interval Δt, the probability that no event occurs is $1-\mu\Delta t+o(\Delta t)$.
4. In the intervals Δt, the probability that 2 or more
   events occur is given by $o(\Delta t)$.

Using the above four assumptions, we can show that of probability P(t) operate during time interval t follow the
Poisson Distribution. Let $P_k(\tau)$ be the probability that
k events occur during a time period of length τ. Then,

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$$P_0(t+\Delta t) = P_0(t)P_0(\Delta t) = P_0(t)[1-\mu\Delta t + o(\Delta t)]$$

$$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\mu P_0(t) + \frac{o(\Delta t)}{\Delta t}$$

$$\frac{dP_0(t)}{dt} = -\mu P_0(t)$$

$$\ln P_0(t) = -\mu t$$

$$P_0(t) = e^{-\mu t} \qquad \text{since } P(0)=1 \text{ as the initial condition}$$

$$P_k(t+\Delta t) = P_k(t)P_0(\Delta t) + P_{k-1}(t)P_1(\Delta t)$$

$$= P_k(t)[1-\mu\Delta t + o(\Delta t)] + P_{k-1}(t)[\mu\Delta t + o(\Delta t)],$$

$$\frac{P_k(t+\Delta t)-P_k(t)}{\Delta t} = -\mu P_k(t) + \mu P_{k-1}(t) + \frac{o(\Delta t)}{\Delta t}[P_k(t)+P_{k-1}(t)]$$

Thus, $$\frac{d(P_k(t))}{dt} = -\mu P_k(t) + \mu P_{k-1}(t)$$

$$\text{For } k=1 \quad P_1(t) = e^{-\mu t}(\mu t)$$

$$\text{For } k=2 \quad P_2(t) = e^{-\mu t}\frac{(\mu t)^2}{2!}$$

$$\text{For } k=n \quad P_n(t) = e^{-\mu t}\frac{(\mu t)^n}{n!}, \qquad n \geq 0.$$

Thus we see that the assumption (1)-(4) given above describe a poisson law. Usually we use poisson distribution to approximation of binomial distribution when p is very small and n is comparatively large. As seen from the above discussion it also arises when we consider a sequence of random events occurring in time or space.

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