SPECIFIC FORMS OF 3×3 $PT-$SYMMETRIC HAMILTONIAN*

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ABSTRACT

The specific forms of PT-symmetric Hamiltonians in 3×3 quantum system are studied in this paper. Depending on the relationship between non-Hermitian and Hermitian matrices, a special property of the 3×3 Hamiltonian satisfying PT symmetry is proved, and then specific forms of 3×3 non-Hermitian but PT symmetric Hamiltonian are presented.

Keywords: PT symmetry; Hamiltonian; Hermitian matrix

0 Introduction

In 1998, Bender C. M. et al.[1] put forward PT symmetric quantum theory, which pointed out that non-Hermitian Hamiltonians had real eigenvalues provided they respect unbroken PT symmetry. Although the observables are represented by Hermitian operators in classical quantum system, non-Hermitian observables also play vital roles in physics [1−13].

PT symmetry is refers to the parity-time symmetry, where P and T stand for parity and time reversal respectively.

In quantum mechanics, $\hat{x}$ and $\hat{p}$ stand for coordinate operator and momentum operator, respectively. Their algorithm is as follows [1]:

$$(\hat{x}f)(x,t) = xf(x,t) \quad , \qquad (\hat{p}f)(x,t) = -i\frac{\partial}{\partial x} f(x,t) .$$

If an operator P satisfies the following equality

$$P\hat{x}P = -\hat{x}, \qquad P\hat{p}P = -\hat{p}, \tag{1}$$

Then P is called parity operator (or space inversion operator) [1], in short operator P. Obviously, it is a linear operator. If operate T satisfies

$$T\hat{x}T = -\hat{x}, \qquad T\hat{p}T = -\hat{p}, \qquad TiT = -i, \tag{2}$$

where $i = \sqrt{-1}$, then T is called time reversal operator [12], in short operator T.

Obviously, it is a conjugate-linear operator.

If H is a $n\times n$ matrix satisfying

$$H = H^{PT}, \tag{3}$$

where $H^{PT} = (PT)H(PT)$, then we say that H is PT-symmetric.

By the definition of operator T, time reversion operator is anti-linear, namely conjugate linear, therefore, it can be divided into the following two categories in general:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \quad \text{or} \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\bar{x} \\ -\bar{y} \end{pmatrix} \tag{4}$$

where $\bar{x}$ stands for the conjugate of x. We denote the above two kinds of operator T as T_1 and T_2 . Obviously, $T_1^2 = T_2^2 = I$ (unit operator).

This paper mainly discusses the specific forms of PT -symmetric Hamiltonians in 3×3 quantum system.

1 Preliminary

We begin with the relationship between non-Hermitian matrix and Hermitian matrix[11], and then prove a special property of 3×3 Hamiltonian satisfying PT symmetry.

Lemma 1. [11] Every non-Hermitian matrix N can be expressed by two Hermitian matrices as follows:

$$N = \frac{1}{2}(H_1 + iH_2), \tag{5}$$

where H_1, H_2 are both Hermitian matrix, which are not zero matrices.

Obviously, the transpose conjugate matrix $N^\dagger$ of N has the following expression

$$N^\dagger = \frac{1}{2}(H_1 - iH_2). \tag{6}$$

and H_1, H_2 must be the following forms:

$$H_1 = \begin{pmatrix} a_1 & b_1 & c_1 \\ \overline{b_1} & d_1 & e_1 \\ \overline{c_1} & \overline{e_1} & f_1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} a_2 & b_2 & c_2 \\ \overline{b_2} & d_2 & e_2 \\ \overline{c_2} & \overline{e_2} & f_2 \end{pmatrix} \tag{7}$$

where $a_1, d_1, f_1, a_2, d_2, f_2$ are real numbers, $b_1, c_1, e_1, b_2, c_2, e_2$ are complex numbers. It is from (5) that any 3×3 non-Hermitian matrix H_N can be represent as follows:

$$H_N = \begin{pmatrix} \frac{a_1 + ia_2}{2} & \frac{b_1 + ib_2}{2} & \frac{c_1 + ic_2}{2} \\ \frac{\overline{b_1} + i\overline{b_2}}{2} & \frac{d_1 + id_2}{2} & \frac{e_1 + ie_2}{2} \\ \frac{\overline{c_1} + i\overline{c_2}}{2} & \frac{\overline{e_1} + i\overline{e_2}}{2} & \frac{f_1 + if_2}{2} \end{pmatrix}, \tag{8}$$

where $a_2, b_2, c_2, d_2, e_2, f_2$ do not equal zero simultaneously.

Lemma 2. Assuming that H is a Hamiltonian of 3×3 quantum system, if H meets PT symmetry, no matter $T = T_1$ or $T = T_2$, for same operator P, they all have $P\overline{H} = HP$.

Proof suppose that

$$H = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}, a, b, c, d, e, f, g, h, k \in £ ,$$

and H meet PT -symmetry.

If $T = T_1$, then $PT_1H = HPT_1$, hence

$$PT_1HT_1 = HPT_1^2 = HP \tag{9}$$

For any $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in C^2$,we have

$$T_1HT_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T_1\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}\begin{pmatrix} \overline{x} \\ \overline{y} \\ \overline{z} \end{pmatrix} = T_1\begin{pmatrix} a\overline{x}+b\overline{y}+c\overline{z} \\ d\overline{x}+e\overline{y}+f\overline{z} \\ g\overline{x}+h\overline{y}+k\overline{z} \end{pmatrix}$$

$$= \begin{pmatrix} \overline{a}x+\overline{b}y+\overline{c}z \\ \overline{d}x+\overline{e}y+\overline{f}z \\ \overline{g}x+\overline{h}y+\overline{k}z \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} & \overline{c} \\ \overline{d} & \overline{e} & \overline{f} \\ \overline{g} & \overline{h} & \overline{k} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overline{H}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

So, $T_1HT_1 = \overline{H}$, and put it into (9), we have

$$P\overline{H} = HP \tag{10}$$

Similarly, if $T = T_2$, we have

$$PT_2HT_2 = HPT_2^2 \tag{11}$$

$$T_1HT_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T_1\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}\begin{pmatrix} -\overline{x} \\ -\overline{y} \\ -\overline{z} \end{pmatrix} = T_1\begin{pmatrix} -\left(a\overline{x}+b\overline{y}+c\overline{z}\right) \\ -\left(d\overline{x}+e\overline{y}+f\overline{z}\right) \\ -\left(g\overline{x}+h\overline{y}+k\overline{z}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \overline{a}x+\overline{b}y+\overline{c}z \\ \overline{d}x+\overline{e}y+\overline{f}z \\ \overline{g}x+\overline{h}y+\overline{k}z \end{pmatrix} = \begin{pmatrix} \overline{a} & \overline{b} & \overline{c} \\ \overline{d} & \overline{e} & \overline{f} \\ \overline{g} & \overline{h} & \overline{k} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \overline{H}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Namely $T_2HT_2 = \overline{H}$, so $P\overline{H} = HP$.

Lemma 3. [12] In finite dimensional space, any operator P, which is commutate to operator T, is a real matrix.

According to the above lemmas, we have established the forms of operator P in 3×3 quantum system[12], in this paper we choose the following form of operator P:

$$P=\begin{pmatrix} -z & 0 & \sqrt{1-z^2} \\ 0 & 1 & 0 \\ \sqrt{1-z^2} & 0 & z \end{pmatrix}, \quad 1 \geq |z| \in \mathbb{R} \tag{12}$$

2 $PT-$symmetric Matrix in 3×3 Quantum System

In this section, let operator T be complex conjugate operator, operator P take form (12). We then present the concrete form of non-Hermitian Hamiltonian H_N which satisfies the PT symmetry in 3×3 quantum system.

It follows from Lemma 1 that any 3×3 non-Hermitian matrix H_N can be represented as (8). If H_N satisfies PT symmetry, then we can calculate the following two quantities:

$$P\overline{H_N}=\begin{pmatrix}-z & 0 & \sqrt{1-z^2}\\ 0 & 1 & 0\\ \sqrt{1-z^2} & 0 & z\end{pmatrix}\begin{pmatrix}\frac{a_1-ia_2}{2} & \frac{\overline{b_1}-i\overline{b_2}}{2} & \frac{\overline{c_1}-i\overline{c_2}}{2}\\ \frac{b_1-ib_2}{2} & \frac{d_1-id_2}{2} & \frac{\overline{e_1}-i\overline{e_2}}{2}\\ \frac{c_1-ic_2}{2} & \frac{e_1-ie_2}{2} & \frac{f_1-if_2}{2}\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}-z(a_1-ia_2)+\sqrt{1-z^2}(c_1-ic_2) & -z\left(\overline{b_1}-i\overline{b_2}\right)+\sqrt{1-z^2}(e_1-ie_2) & -z\left(\overline{c_1}-i\overline{c_2}\right)+\sqrt{1-z^2}(f_1-if)\\ b_1-ib_2 & d_1-id_2 & \overline{e_1}-i\overline{e_2}\\ \sqrt{1-z^2}(a_1-ia_2)+z(c_1-ic_2) & \sqrt{1-z^2}\left(\overline{b_1}-i\overline{b_2}\right)+z(e_1-ie_2) & \sqrt{1-z^2}\left(\overline{c_1}-i\overline{c_2}\right)+z(f_1-if)\end{pmatrix}$$

(13)

$$H_NP=\begin{pmatrix}\frac{a_1+ia_2}{2} & \frac{b_1+ib_2}{2} & \frac{c_1+ic_2}{2}\\ \frac{\overline{b_1}+i\overline{b_2}}{2} & \frac{d_1+id_2}{2} & \frac{e_1+ie_2}{2}\\ \frac{\overline{c_1}+i\overline{c_2}}{2} & \frac{\overline{e_1}+i\overline{e_2}}{2} & \frac{f_1+if_2}{2}\end{pmatrix}\begin{pmatrix}-z & 0 & \sqrt{1-z^2}\\ 0 & 1 & 0\\ \sqrt{1-z^2} & 0 & z\end{pmatrix}=$$

$$\frac{1}{2}\begin{pmatrix}-z(a_1+ia_2)+\sqrt{1-z^2}(c_1+ic_2) & -z\left(\overline{b_1}+i\overline{b_2}\right)+\sqrt{1-z^2}(e_1+ie_2) & -z\left(\overline{c_1}+i\overline{c_2}\right)+\sqrt{1-z^2}(f_1+if)\\ b_1+ib_2 & d_1+id_2 & \overline{e_1}+i\overline{e_2}\\ \sqrt{1-z^2}(a_1+ia_2)+z(c_1+ic_2) & \sqrt{1-z^2}\left(\overline{b_1}+i\overline{b_2}\right)+z(e_1+ie_2) & \sqrt{1-z^2}\left(\overline{c_1}+i\overline{c_2}\right)+z(f_1+if)\end{pmatrix}$$

(14)

Note that $P\overline{H_N}=H_NP$ by Lemma 2, so from (13) and (14) we have

$$\begin{cases}-z(a_1-ia_2)+\sqrt{1-z^2}(c_1-ic_2)=-z(a_1+ia_2)+\sqrt{1-z^2}(c_1+ic_2)\\ -z\left(\overline{b_1}-i\overline{b_2}\right)+\sqrt{1-z^2}(e_1-ie_2)=b_1+ib_2\\ -z\left(\overline{c_1}-i\overline{c_2}\right)+\sqrt{1-z^2}(f_1-if_2)=z(c_1+ic_2)+\sqrt{1-z^2}(a_1+ia_2)\\ b_1-ib_2=-z\left(\overline{b_1}+i\overline{b_2}\right)+\sqrt{1-z^2}(e_1+ie_2)\\ d_1-id_2=d_1+id_2\\ \overline{e_1}-i\overline{e_2}=z(e_1+ie_2)+\sqrt{1-z^2}\left(\overline{b_1}+i\overline{b_2}\right)\\ z(c_1-ic_2)+\sqrt{1-z^2}(a_1-ia_2)=-z\left(\overline{c_1}+i\overline{c_2}\right)+\sqrt{1-z^2}(f_1+if_2)\\ z(e_1-ie_2)+\sqrt{1-z^2}\left(\overline{b_1}-i\overline{b_2}\right)=\overline{e_1}+i\overline{e_2}\\ z(f_1-if_2)+\sqrt{1-z^2}\left(\overline{c_1}-i\overline{c_2}\right)=z(f_1+if_2)+\sqrt{1-z^2}\left(\overline{c_1}+i\overline{c_2}\right)\end{cases} \quad (15)$$

We can easily get that $c_2 \in$ ¡ and (16) from the fourth equality in (15),

$$\begin{cases} c_2 \in \mathbb{R},\ a_2 + f_2 = 0,\ d_2 = 0 \\ \sqrt{1-z^2}\,(f_1 - a_1) = (z c_1 + \overline{c_1}) \\ \sqrt{1-z^2}\, e_1 = \overline{z}\, b_1 + b_1 \\ \sqrt{1-z^2}\, e_2 = \overline{z}\, b_2 - b_2 \\ \sqrt{1-z^2}\, \overline{b_1} = \overline{e_1} - z e_1 \\ \sqrt{1-z^2}\, \overline{b_2} = -\overline{e_2} - z e_2 \end{cases}, \quad c_2 \in \mathbb{R}, a_2 + f_2 = 0, d_2 = \qquad (16)$$

In order to fully ensure the relationship between various parameters in (15) , and further specific the forms of H_N, we analyze (15) in three cases: (I) $z=0$; (II) $z=\pm 1$; (III) $z \notin \{-1,0,1\}$.

(I) If $z=0$, then $a_1 = f_1, e_1 = b_1, e_2 = -b_2$, so

$$H_N = \frac{1}{2}\begin{pmatrix} a_1+ia_2 & b_1+ib_2 & c_1+ic_2 \\ \overline{b_1}+i\overline{b_2} & d_1 & b_1 - ib_2 \\ \overline{c_1}+ic_2 & \overline{b_1} - i\overline{b_2} & a_1 - ia_2 \end{pmatrix}, \qquad (17).$$

For example, let $a_1 = c_2 = 0, a_2 = 2, b_1 = -b_2 = c_1 = i, d_1 = 1$, then we can take H_N as follows,

$$H_N = \frac{1}{2}\begin{pmatrix} 2i & i+1 & i \\ -i+1 & 1 & i-1 \\ -i & -i-1 & -2i \end{pmatrix}. \qquad (18)$$

(II) If $\sqrt{1-z^2}=0$, namely $z=\pm 1$.

If $z=1$, $b_2, e_1 \in \mathbb{R}$ and the real parts of b_1, c_1, e_2 are all zeros, so

$$H_N = \frac{1}{2}\begin{pmatrix} a_1+ia_2 & b_1+ib_2 & c_1+ic_2 \\ -b_1+ib_2 & d_1 & e_1+ie_2 \\ -c_1+ic_2 & e_1 - ie_2 & f_1 - ia_2 \end{pmatrix}, \qquad (19)$$

For example, let $a_1 = c_2 = d_1 = e_1 = f_1 = 1, a_2 = 2, b_2 = 0, c_1 = i$, then we can take H_N as follows,

$$H_N = \frac{1}{2}\begin{pmatrix} 1+2i & i & i+1 \\ -i & 1 & 0 \\ 0 & 2 & 1-2i \end{pmatrix}. \qquad (20)$$

If $z=-1$, $b_1, b_2, e_2 \in \mathbb{R}$ and the real parts of c_1, e_1 are zeros, then

$$H_N=\frac{1}{2}\begin{pmatrix} a_1+ia_2 & b_1+ib_2 & c_1+ic_2 \\ b_1+ib_2 & d_1 & e_1+ie_2 \\ -c_1+ic_2 & -e_1+ie_2 & f_1-ia_2 \end{pmatrix} \tag{21}$$

For example, let $a_1=a_2=b_1=b_2=d_1=f_1=1, c_2=e_2=0, e_1=2i$, then we can take H_N as follows,

$$H_N=\frac{1}{2}\begin{pmatrix} 1+i & 1+i & i \\ 1+i & 1 & 2i \\ -i & -2i & 1-i \end{pmatrix}. \tag{22}$$

(III) If $z\notin\{-1,0,1\}$, we have $e_1=\frac{1}{\sqrt{1-z^2}}\left(z\overline{b_1}+b_1\right)$, $e_2=\frac{1}{\sqrt{1-z^2}}\left(z\overline{b_2}-b_2\right)$, then (16) can be changed into

$$H_N=\frac{1}{2}\begin{pmatrix} a_1+ia_2 & b_1+ib_2 & c_1+ic_2 \\ \overline{b_1}+i\overline{b_2} & d_1 & \frac{\left(z\overline{b_1}+b_1\right)}{\sqrt{1-z^2}}+\frac{i\left(z\overline{b_2}-b_2\right)}{\sqrt{1-z^2}} \\ \overline{c_1}+ic_2 & \frac{\left(zb_1+\overline{b_1}\right)}{\sqrt{1-z^2}}+\frac{i\left(zb_2-\overline{b_2}\right)}{\sqrt{1-z^2}} & f_1-ia_2 \end{pmatrix} \tag{23}$$

with $a_1, f_1, c_1, \overline{c_1}$ satisfied $\sqrt{1-z^2}\left(f_1-a_1\right)=z\left(c_1+\overline{c_1}\right)$.

3 CONCLUSION

This paper mainly discussed the concrete form of non-Hermitian Hamiltonian satisfying PT symmetry condition in 3×3 quantum system. Depending on the relationship between the non-Hermitian and Hermitian matrices, we first established a special property of the Hamiltonian satisfying PT symmetry, $P\overline{H}=HP$, then we analyzed the specific forms of the non-Hermitian under different conditions.

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