BROUWER'S FIXED-POINT THEOREM IN PLANE GEOMETRY

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ABSTRACT

This study is about the proof of the theorem known as the first basic Fixed-Point Theorem found by L. E. J. Brouwer between the year 1909 and 1913 in plane geometry. As known, there have been other studies on fixed-point after Brouwer, other theorems were presented, proven and brought into the literature. When we look at these theorems, we see that they are usually used by fields such as analysis and topology in Turkey and abroad as significant tools with applications. In this article on the other hand, we will prove Brouwer's theorem that **'C** being a unit sphere on R^n there is a fixed-point for the function f: C → C in continuous transformation [7],' using concepts of expansions/compressions, pushes and rotations in plane geometry.

Keywords: Fixed-Point Theorem, Plane Geometry, Expansion/Compression, Push, Rotation

1. INTRODUCTION and DEFİNİTİONS

Firstly, we would like to begin with a short history of fixed point theory as it can be also seen in the works of Lennard and Nezir [14,15]. Researches on fixed point theory initiated in 1912 by L.E.J. Brouwer 's result [2]. Brouwer showed that for $n \in N$, for C the set of the closed unit ball of R^n every norm-to-norm continuous function $f: C \to C$ has a fixed point. Later, in 1930 Schauder [16] generalized Brouwer's result to every compact convex subset of R^n This class of continuous functions was very large and so researches wanted to work on smaller classes of the functions and researched fixed points of those. Then, in 1922, the well-known fixed point theorem arrived. Banach gave his theorem so called Banach Contraction theorem [1] as follows: if (X, d) is a complete metric space and $f: X \to X$ is a strict contraction for the metric $d = d_{\|\cdot\|}$ generated by the norm, then f has a unique fixed point. Later, more technical theorems were proven by other researches. Indeed, 1965 was very efficient year for the fixed point theory and in this year, there were three big results for this field by Browder, Gothe and Kirk.

In 1965, Browder [3] proved : (For every closed, bounded, convex (non-empty) subset C of a Hilbert space (X, $\|.\|$), for all nonexpansive mappings T: C → C (i.e., $\|Tx - Ty\| \le \|x - y\|$, for all x, y), T has a fixed point in C.) Soon after, also in 1965, Browder and Göhde (independently) generalized the result to uniformly convex Banach space (X, $\|.\|$), e.g., X= L^p, 1< p < ∞ , with its usual norm $\|.\|_p$.

Later in 1965,Kirk [11] further generalised to all reflexive Banach space X with normal structure: those spaces such that all-trivial closed, bounded, convex sets C have a smaller radius than diameter.

Spaces (X, $\|.\|$) with the property of Browder became known as spaces with the "fixed point property for nonexpansive mappings" (FPP(n.e)).After 48 years, it remains an open question as to whether or not every reflexive Banach space $(X, \|.\|)$ has the fixed point property for nonexpansive mappings.

In 1960 Kutuzov B.V. mentioned the transformation of shapes and stable points of these shapes in his book named Geometry, studies in Mathematics Vol. IV [13]
Again, in 1973 S.R. Clemens published an article named Fixed Point Theorems in Euclidean Geometry. In this article he explained how Euclidean Geometry Fixed Point works. He also showed Menelaus and Seva Theorems are good application of Fixed Point Theorem. [4]
In our study, we stressed on examples of plane geometry of L.E.J. Brouwer Fixed Point Theorem. Our purpose is to bring plane geometry into the research conducted on fixed-points and open up a new field.

1.1. Definition (Fixed-Point): Let X be a non-empty set, and $T: X \to X$ a function. If there is at least one $x \in X$ that satisfies $T(x) = x$, x is a fixed point of the function T. All $x \in X$ satisfying this equality are fixed-points of the function T [15].

1.2. Definition (Compressive Transformation): Let (X, d) be a metric space and $T: X \to X$ any transformation (function). If for all $x, y \in X$, there is a number α $(0 \leq \alpha < 1)$ satisfying $d(Tx, Ty) \leq \alpha.d(x, y)$, T is called a **compressive transformation** [15].

1.3. Definition (Closed Unit Sphere): The subset $\overline{B}_d(x, 1) \subset R^n$ for $\forall n \in N$ is called a closed unit sphere of radius 1 with the center x in R^n.
It is defined as $\overline{B}_d(x, 1) = \{y \in R^n: d(x, y) \leq 1, x \in R^n\}$ [8].

1.4. Definition (Expansion-Dilation): The similarity transformation mapping parallel lines to parallel lines and preserving direction is called a dilation. This transformation is also a collineation, mapping points to points, lines to lines, and planes to planes [5].

1.5. Definition:

1. Expansion/compression transformation for a point (x, y) in a plane is defined as $G_k(x, y) = (kx, ky)$ while $k \in [0, 1)$ [15].
2. Counterclockwise rotation of as point (x, y) in a plane by θ is defined as $R_\theta(x, y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \begin{pmatrix} x \\ y \end{pmatrix}$ [9].
3. Pushing a point (x, y) in a plane is defined as $T_{\vec{V}}(x, y) = (x + a, y + b)$ while $\vec{V}(a, b)$ is the pushing vector [9].

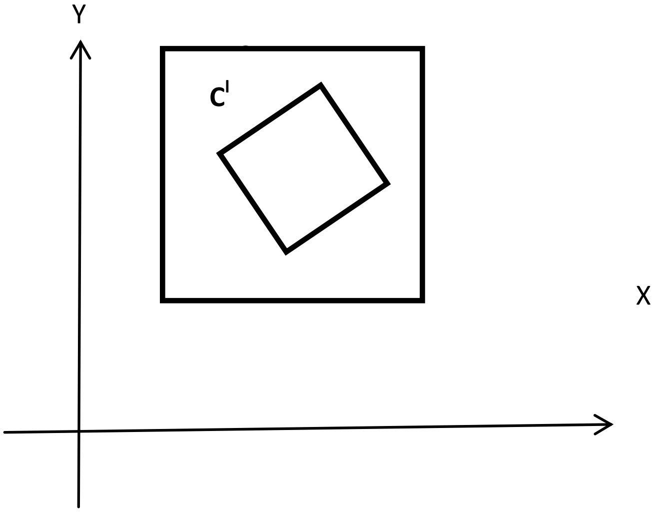
1.1. Theorem (Brouwer's Fixed-Point Theorem):
While any $C = \overline{B}_d(x, 1) \subset R^n$ having $x \in R^n$ for $\forall n \in N$ is a closed unit sphere, the continuous transformation $T: C \to C$ has a fixed-point. That is, any continuous transformation mapping R^n's closed unit sphere to itself has a fixed-point [7].

1.1. Proof: For the continuous transformation $f: [a, b] \to [a, b]$ while $C = [a, b]$, there exists $x \in [a, b]$ satisfying $f(x) = x$.
If we take $F(x) = x - f(x)$, and let $F(a) = a - f(a) \leq 0$ and $F(b) = b - f(b) \geq 0$, according to the intermediate value theorem, there is at least one $x \in [a, b]$ such that $F(x) = 0$. If $F(x) = 0$, then $x - f(x) = 0$, meaning $f(x) = x$.
Q.E.D. [10].

2. MAİN RESULTS

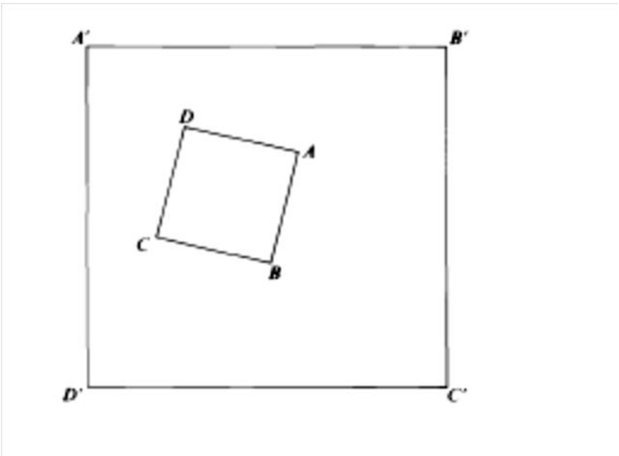
It is possible to provide various proofs of Brouwer's Fixed-Point Theorem using analysis and topology. In our study, we will provide one proof from analysis and demonstrate other proofs over plane geometry. The proof of plane geometry mentioned above stated in four diffent ways as main result.

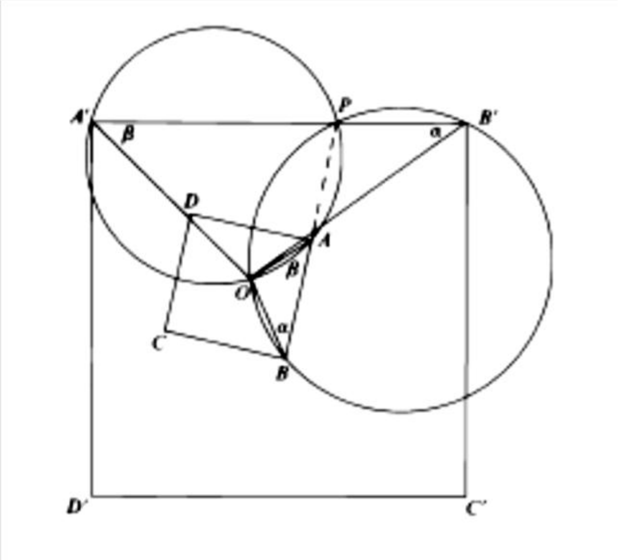
Proof 2.1: Let us chose our C closed sphere as a square and define our continuous transformation f as f: $C \rightarrow C^{l}$, mapping the closed square to another square. Let us show that this transformation has at least one fixed-point.



The transformation, while mapping points from the bigger square to the smaller square, leaves at least one point fixed. We will first try to demonstrate how such a fixed-point is found, and then try to see whether such a transformation can be explained by an equation.

Proof 2.2:

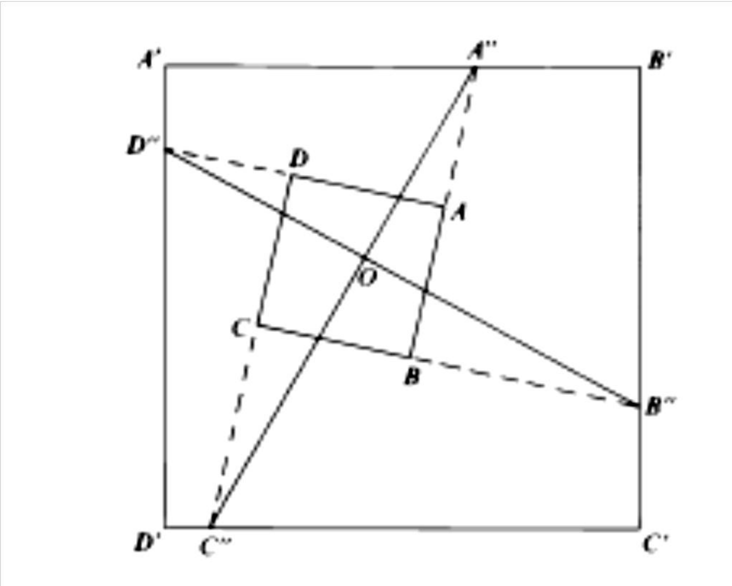




Let us demonstrate our proof with the theorem that the compressive transformation has one fixed-point. This theorem is a special case of Brouwer's Fixed-Point Theorem, and it is proven by directly repeating the transformation. In this problem, the transformation gives us a series of smaller and smaller squares represented inside bigger ones. Therefore, we obtain a series of intertwined areas converging to a size of zero. The limit and the intersection of these intertwined areas is a fixed-point. As we know from plane geometry, any two given squares are similar. So, there is a similarity making a square into another square.

Assuming AB intersects with $A'B'$ on the point P and we draw two circles passing from points A, P, A' and points B,P, B'. These circles intersect on the fixed-point O. To see this, note that the angles denoted as α are equal as they are equal to the half of the angle of the arc *PO*. Angles denoted as β are also equal, as they are both the supplementary angle of the angle *PAO*. Thus, the triangle OAB and the triangle OA^1B^1 are similar triangles. From the similarity of these two triangles, A^1B^1 is rotated over the angle AOA' on the axis 'O', and places the ABCD square inside, while overlapping them with an expansion that can be represented with the k = $\frac{A'B'}{AB}$ ratio [12].

Proof 2.3:

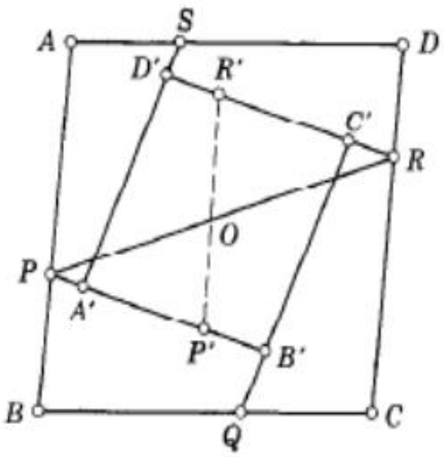


For addition of extended edges, we find the intersections of AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, and finally DA and $D'A'$.

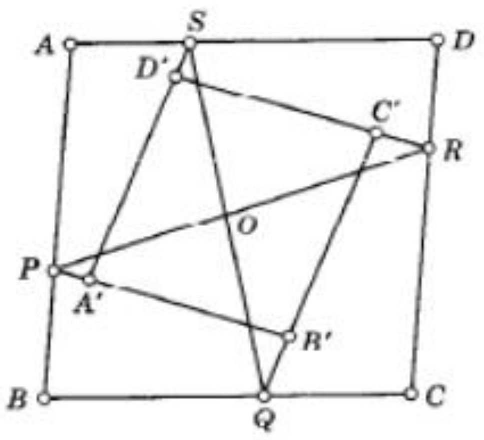
Then we show these points as A'', B'', C'', D'' and O''. Therefore, the fixed-point is the intersection point of the similarity, meaning the intersection of *A"C" and B"D"*.

The shape occurring over the lines AB and CD and the point 'O', and the shape occurring over the lines $A'B'$ and $C'D'$ and the point 'O' are similar shapes. Therefore, an expansion including a line passing through 'O' from AB to CD, also includes the lines from $A'B'$ to $C'D'$ and from A'' to CD. Thus 'O', A'' and C'' are in the same direction. Similarly, 'O', B'' and D'' are in the same direction [12].

Proof 2.4:



Let our continuous transformation T transform the square ABCD to the square $A'B'C'D'$. Let P, Q, R, S be non-coincident points on the edges. This similarity transformation of:
T(A) = A'
T(B) = B'
T(C) = C'
T(D) = D'
maps P on AB to *P'* on *A'B'*, and R on CD to *R'* on *C'D'*. The intersection of the PR and *P'R'* line segments is the point O. While there is a similarity transformation ratio of *P'R'* = k.PR here, this ratio may be defined as $\frac{OP}{OR} = \frac{OP'}{OR'}$ and this transformation leaves the point O fixed. Thus T(O) = O.
This point O is also the intersection of the line segments PR and QS as shown below.



It can also be demonstrated that the following similarity ratios exist in the shape above.

$\frac{PA}{PB} = \frac{PS}{PQ} = \frac{PA'}{PB'}$ and

$\frac{RC}{RD} = \frac{PQ}{RS} = \frac{RC'}{RD'}$

With these similarity ratios, the point shown as T(O) = O is the fixed-point of the continuous transformation T [5].

Proof 2.5:

Let us use the concepts of expansion/compression, rotation and pushing in plane geometry in this proof of ours.

Stage 1:

Let us take a square ABCD and apply the expansion/compression transformation defined as $G_k(x, y) = (kx, ky)$, $k \in [0, 1)$ on this square, resulting in the square $A^1B^1C^1D^1$.

Stage 2:

Let us take the corner points of $A^1B^1C^1D^1$ and apply the rotation transformation $R_\theta(x, y) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$ = (xcosθ - ysinθ, xsinθ + ycosθ), which is the rotation of a point (x, y) in a plane by θ counterclockwise, to obtain the square $A^{II}B^{II}C^{II}D^{II}$.

Stage 3:

Let us apply the pushing transformation defined as $T_{\vec{V}}$ (x, y) = (x + a, y + b) with the pushing vector $\vec{V}$(a, b) on the corner points of the square $A^{II}B^{II}C^{II}D^{II}$ and obtain the square $A^{III}B^{III}C^{III}D^{III}$.

There is such a fixed-point contained in the square ABCD and the last obtained square $A^{III}B^{III}C^{III}D^{III}$ that, it had not changed place after these three transformations.

Applying these three transformations can be defined as the composite function

$(T_{\vec{V}} o R_\theta o G_k)(x, y) = T_{\vec{V}}(R_\theta(G_k(x,y)))$. This transformation has a fixed point.

So, the equation $T_{\vec{V}}(R_\theta(G_k(x, y))) = (x, y)$ is solvable. Let us obtain our function to solve for equality.

Let us put the equation $G_k(x, y) = (kx, ky)$ into place and obtain $T_{\vec{V}}(R_\theta(kx, ky))$. Let us put (kxcosθ - kysinθ, kxsinθ + kycosθ) in place of $R_\theta(kx, ky)$.

Our statement is now $T_{\vec{V}}$ (kxcosθ - kysinθ, kxsinθ + kycosθ). Computing for this statement, we obtain the equation ($T_{\vec{V}}$ oR_θoG_k)(x, y) = (a + kxcosθ - kysinθ, b + kxsinθ + kycosθ) = (x, y). We have obtained two first degree equations and their solution is simple.
Let us organize the equations a + kxcosθ - kysinθ = x and b + kxsinθ + kycosθ = y.

a = (1 - kcosθ)x + kysinθ ……….(1)

b = (1 - kcosθ)y - kxsinθ ……….(2)

These equations can be solved with the elimination method. The point (x, y) obtained with this solution is the fixed-point of the transformation defined between two squares [6].

EXAMPLE:

This section will demonstrate that the composite transformation obtained above is a transformation that has a fixed-point for any random square.

Let A($8\sqrt{2}$, $8\sqrt{2}$), B($8\sqrt{2}$, $-8\sqrt{2}$), C($-8\sqrt{2}$, $-8\sqrt{2}$) and D($-8\sqrt{2}$, $8\sqrt{2}$) be the corner points of the square ABCD.

Stage 1:

Let us apply the expansion/compression transformation of: $G_{\frac{1}{8}}$ (x, y) = ($\frac{1}{8}$x, $\frac{1}{8}$x), transforming the square ABCD to the square A'B'C'D'.

We obtain the points A'($\sqrt{2}$, $\sqrt{2}$), B'($\sqrt{2}$, $\sqrt{2}$), C'($\sqrt{2}$, $\sqrt{2}$) and D'($\sqrt{2}$, $\sqrt{2}$).

Stage 2:

Let us rotate the square A'B'C'D' counterclockwise around the origin by 45°.

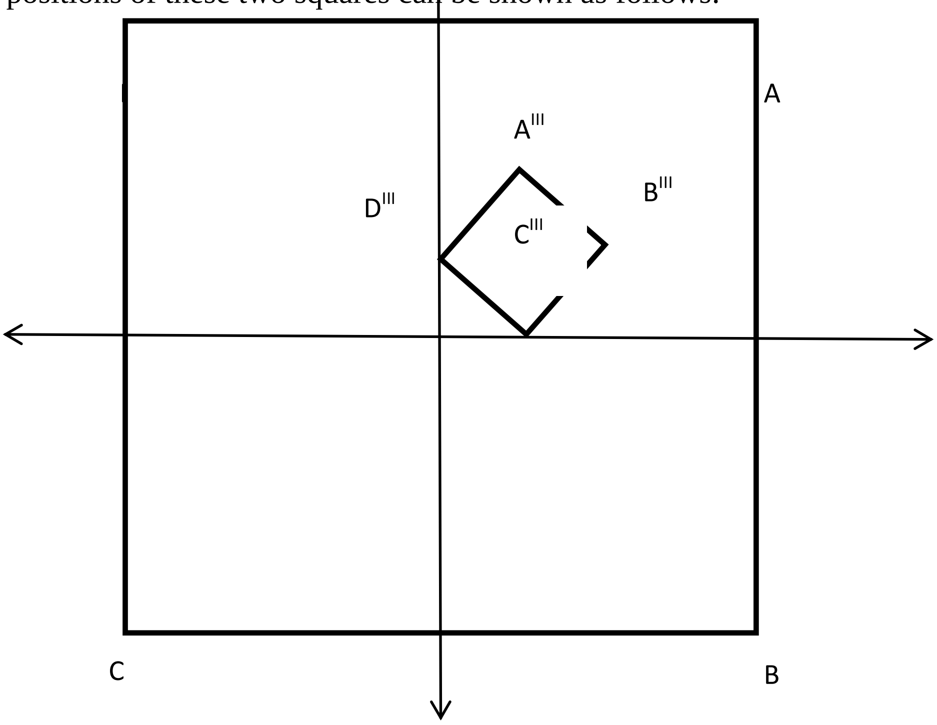
By substituting 45° for θ in the transformation R_θ(x, y)= $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} . \begin{pmatrix} x \\ y \end{pmatrix}$, we obtain the square A''B''C''D''.

A''(0, 2), B''(2 ,0), C''(0, -2) and D''(-2, 0) are the corner points of the square A''B''C''D''.

Stage 3: Let us apply the transformation of pushing the corners of the square A''B''C''D'' to the right by 2 units and upwards by 2 units, obtaining the square A'''B'''C'''D''' as a result. The corner points of this square can be found as:

A'''(2, 4), B'''(4, 2), C'''(2, 0) and D'''(0, 2).

The positions of these two squares can be shown as follows:



If we write down the operations we have done by this stage in terms of a composite function, we obtain the following:
$(T_{\vec{V}}\, oR_\theta oG_k)(x, y) = (a + kx\cos\theta - ky\sin\theta,\ b + kx\sin\theta + ky\cos\theta)$

Putting into place the values $k = \frac{1}{8}$, $(a, b) = (2, 2)$, $\theta = 45^\circ$ and $\sin 45^\circ = \frac{\sqrt{2}}{2}$, $\cos 45^\circ = \frac{\sqrt{2}}{2}$:
We obtain the transformation:
$(T_{\vec{V}}\, oR_\theta oG_k)(x, y) = (2 + \frac{1}{8}x\frac{\sqrt{2}}{2} - \frac{1}{8}y\frac{\sqrt{2}}{2},\ 2 + \frac{1}{8}x\frac{\sqrt{2}}{2} + \frac{1}{8}y\frac{\sqrt{2}}{2})$.
Now let us find the fixed-point by solving the equation $(T_{\vec{V}}\, oR_\theta oG_k)(x, y) = (x, y)$. When the equalities are organized, the following equations are obtained:
$(\sqrt{2} - 16)x - \sqrt{2}\, y = -32$
$\sqrt{2}\, x + (\sqrt{2} - 16)y = -32$
Solving these equations and taking the approximate values of the square rood, we find the following values: x = 1.962 and y = 2.384. Therefore, the fixed-point of this transformation is (x, y) = (1.962, 2.384).

Conclusion: In this study, we have demonstrated that it is possible to prove the Fixed-Point Theorem with concepts of expansion/compression, pushing and rotation in plane geometry. We hope we will make the concept of Fixed-Point a more visualized one and save it from staying as an abstract concept.

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