OPERATOR P IN 2×2 $PT-$SYMMETRIC QUANTUM SYSTEM*

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ABSTRACT

The forms of operator P are discussed, which ensure a known 2×2 Hamiltonian to be $PT-$symmetric, in the premise that the operator T is the complex conjugate operator.

Keywords: $PT-$symmetry; operator P; operator T; Hamiltonian.

INTRODUCTION

In 1998, Bender C. M. et al. [1] put forward PT symmetric quantum theory, which pointed out that non-Hermitian Hamiltonians had real eigenvalues provided they respect unbroken PT symmetry. PT symmetry is refers to the parity-time symmetry, where P stand for parity and T stand for time reversal. Using non-Hermitian PT - symmetric Hamiltonian to describe quantum mechanics theory was not accepted at the beginning, since people always thought that non-Hermitian physical quantities do not have real eigenvalues, which had no practical meaning.

Bessis D. $[2-3]$ had found that the following Hamiltonian have real eigenvalues,

$$H=p^2+x^2+ix^3, \quad (1)$$

but did not explain the reason until Bender C. M. found that (1) met PT symmetry. So far there are lots of results about PT - symmetric theory $[3-11]$. Professor Bender have given some examples of PT - symmetric Hamiltonians, especially the general form of the 2×2 PT - symmetric Hamiltonian:

$$H=\begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}, \quad (2)$$

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where $r, s, \theta \in R$, T takes complex conjugate and $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Naturally, there is a question from the mathematical point of view： is there any other forms of operator P ensuring that the Hamitonian H in (2) is also PT - symmetric where T takes complex conjugate? In this paper, we mainly discuss this problem and give four forms of operator P.

Preliminary

In this section we recall the basic concepts of operator P, operator T and PT symmetry.

In quantum mechanics, $\hat{x}$ and $\hat{p}$ stand for coordinate operator and momentum operator, respectively. Their algorithm is as follows[10]:

$$(xf)(x,t) = xf(x,t) \quad , \tag{3}$$

$$(\hat{p}f)(x,t) = -i\frac{\partial}{\partial x} f(x,t) . \tag{4}$$

If operate P satisfies the following equality

$$P\hat{x}P = -\hat{x}, \qquad P\hat{p}P = -\hat{p} , \tag{5}$$

Then P is called parity operator (or space inversion operator) [10], in short operator P. Obviously, it is a linear operator. If operate T satisfies

$$T\hat{x}T = -\hat{x}, \qquad T\hat{p}T = -\hat{p}, \qquad TiT = -i, \ , \tag{6}$$

where $i = \sqrt{-1}$, then T is called time reversal operator[10], in short operator T. Obviously, it is a.conjugate-linear operator.

From the above definition, it is easy to know that operator P and T are projectors, namely

$$P^2 = T^2 = I , \tag{7}$$

and P commutates to T, namely

$$PT - TP = 0 . \tag{8}$$

So we can also use formula (7) and (8) to determine operator P and T.

If H is a $n \times n$ matrix satisfying

$$H = H^{PT}, \tag{9}$$

where $H^{PT} = (PT)H(PT)$, then we say that H is PT - symmetric.

Denoting $[A,B] = AB - BA$, then formula (9) can be written as follows $[H,PT] = HPT - PTH = 0$ or $HPT = PTH$.

By the definition of operator T, time reversion operator is anti-linear, namely conjugate linear, therefore, it can be divided into the following two categories in general:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \quad \text{or} \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\bar{x} \\ -\bar{y} \end{pmatrix} \tag{10}$$

where $\bar{x}$ stands for the conjugate of x. We denote the above two kinds of operator T as T_1 and T_2. Obviously, $T_1^2 = T_2^2 = I$ （unit operator）.

The forms of Operator P

In this section, let operator T be complex conjugate operator, we then discuss the forms of operator P, which ensures the Hamiltonian H in (2) to be $PT-$symmetric. We begin with some lemmas we have got in [12].

Lemma 1. [12] Assuming that H is a Hamiltonian of 2×2 quantum system, if H meets PT symmetry, no matter $T = T_1$ or $T = T_2$, for same operator P, they all have $P\overline{H} = HP$.

Lemma 2. [12] In finite dimensional space, any operator P, which is commutate to operator T, is a real matrix.

Theorem 1. Let T be complex conjugate operator, if the Hamiltonian H in (2) satisfies PT symmetry, then operator P has the following forms:

1) if $s = 0$, then $P = \begin{pmatrix} 0 & b \\ \frac{1}{b} & 0 \end{pmatrix}$, where $b \neq 0$;

2) if $s \neq 0$, then $P = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$, where $a^2 + b^2 = 1,\ ab = 0$。

Proof. No loss of generality, we only consider the first type of operator T (the second type of operator T can be discussed similarly). From Lemma 2, we assume that

$$P=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad T\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}, \qquad a,b,c,d \in R,$$

We can easily calculate that

$$P\overline{H}=\begin{pmatrix} bs+are^{-\theta i} & as+bre^{\theta i} \\ ds+cre^{-\theta i} & cs+dre^{\theta i} \end{pmatrix}, \tag{11}$$

$$HP=\begin{pmatrix} cs+are^{\theta i} & ds+bre^{\theta i} \\ as+cre^{-\theta i} & bs+dre^{-\theta i} \end{pmatrix}, \tag{12}$$

From Lemma 1, $P\overline{H}=HP$, then we get the following equation group

$$\begin{cases} cs+are^{\theta i}=bs+are^{-\theta i} \\ ds+bre^{\theta i}=as+bre^{\theta i} \\ as+cre^{-\theta i}=ds+cre^{-\theta i} \\ bs+dre^{-\theta i}=cs+dre^{\theta i} \end{cases}, \tag{13}$$

Obviously, $r \neq 0$, $\theta \neq 0$, otherwise the Hamiltonian H in (2) becomes Hermitian, which is not within the scope of our discussion. Note that $P^2=I$, then $bc=1$.

If $s=0$, then from (13) we have $a=d=0$, hence

$$P=\begin{pmatrix} 0 & b \\ \frac{1}{b} & 0 \end{pmatrix},$$

where $b \neq 0$.

If $s \neq 0$, then (13) can be described as follows

$$\begin{cases} r(a-d)e^{-\theta i}=r(a-d)e^{\theta i} \\ r(b-c)e^{-\theta i}=r(b-c)e^{\theta i} \end{cases}, \tag{14}$$

Since $r\theta \neq 0$, then (14) implies that $a=d, b=c$, from $P^2=I$ again we get

$$P=\begin{pmatrix} a & b \\ b & a \end{pmatrix},$$

where $a^2+b^2=1, ab=0$.

Therefore, if $a=0$, then $b=\pm1$; if $b=0$, then $a=\pm1$. That is to say, operator P has the following four forms:

$$P=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad P=\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix},\quad P=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\quad P=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}.$$

In summary, the form of operator P is not unique.

CONCLUSION

In this paper, we have given a general discussion of operator P for a known $PT-$symmetric Hamiltonian in 2×2 quantum system. Under the premise that the operator T is a complex conjugate operator, we present four forms of operator P which make the known Hamiltonian always be $PT-$symmetric.

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