SEMI-MEMBRANE SHELL THEORY OF HYBRID ANISOTROPIC MATERIALS

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ABSTRACT

A semi-membrane shell theory of hybrid anisotropic materials is developed by means of asymptotic integration method starting from a three dimensional elastic element. A very long effective length scale of cylindrical shell is adopted for the formulation, edge effects due to the prescribed boundary condition penetrate differently depending on material orientation of each layer but all within the limit of length scale $(ah)^{1/2}$ where Donnell-Vlasov bending theory is valid. Demonstrated that beyond the limit of edge effective zone, membrane or pseudo-membrane state dominates, it is traditionally named semi-membrane state. Governing equations of semi-membrane theory of cylindrical shell are formulated and the physical interpretation of the theory is described. The theory is very useful to analyze and design long cylindrical shells such as rocket fuel storage tanks, open top oil stage tanks and chimney structures.

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INTRODUCTION

It is known that the semi-membrane shell theory was first formulated by Vlasov, he might have utilized the mathematics but never used the terminology "semi-membrane" at all. Who started to name the terminology and how Vlasov was honored are an interesting issue. When analyzing the cylindrical shells, it is common to utilize known physical parameters such as radius (a) thickness (h) and length (L). Donnell, Vlasov and many other scientists classified when develop the analytical model for circular cylindrical shell as following categories:

Short Effective Length
Intermediate Effective Length
Very long effective Length

Cylindrical shells are used for space shuttles, rocket fuel storage tanks, aircraft fuselages, above ground fuel storage tanks and deep water submarines. Depending on the environment where the shell structures are located, it will take positive or negative pressure, we must assure that the structure should never be collapsed.

The advantages of cylindrical shells are not only its functional capacity and aerodynamic features but also a simple coordinate system for the mechanical analysis compared to spherical or conical shapes. We will first accept and use the classical assumption of Love then compare our theory with more elaborate theories of Reissner and Donnell-Vlasov as shown in the References [1], [2] and [3].

Once we built the analysis for the cylindrical shells we could easily convert to the other shape shells as shown in the References, [8], [17], [16], [19], [20]. However, the mathematics to describe its behavior and characteristics are complicated and it is more of challenge when the materials are anisotropic and combination of different anisotropic materials, hybrid anisotropic shell structures.

Let us first start with three dimensional coordinate system of shell structures.
The system is of longitudinal (X, z), circumferential (φ, θ) and radial (r) as shown in Figure 1 and 2. The original and non-dimensional coordinates as shown in the figures are used to allow an asymptotic integration process. According to the exact three-dimensional theory of elasticity, a shell element is considered as a volume element. All possible stresses and strains are assumed to exist and no simplifying assumptions are allowed in the formulation of the theory. We therefore allow for six stress components, six strain components and three displacements as indicated in the following equation:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad i, j = 1, 2, 3 \qquad k, l = 1, 2 \tag{1}$$

where σ_{ij} and ε_{kl} are stress and strain tensors respectively and C_{ijkl} are elastic moduli.
There are thus a total of fifteen unknowns to solve for in a three dimensional elasticity problem. On the other hand, the equilibrium equations and strain displacement equations can be obtained for a volume element and six generalized elasticity equations can be used. A total of fifteen equations can thus be formulated and it is basically possible to set up a solution for a three-dimensional elasticity problem. It is however very complicated to obtain a unique solution which satisfies both the above fifteen equations and the associated boundary conditions. This led to the development of various theories for structures of engineering interest. A detailed description of classical shell theory can be found in various references [1] through [13].

The cylindrical shell theory that we are concerned is among the three classifications of above is very long effective length shell which is well compared to Donnell-Vlasov theory. As shown in Figure 4, the edge effective zone due to the prescribed edge boundary condition represented by curved pattern is limited within close distance from the end and after the zone the deformation is nearly linear with respect to the longitudinal axis, which is very close to membrane analysis. It is more distinct for a cylindrical shell of one end fixed or hinged boundary condition and the other free open. That is semi-membrane status, we will now mathematically formulate the governing equations.

According to Calladine, Vinson and Chung, References [18], [21], [22] respectively, we pick and choose the longitudinal (L) and circumferential (ℓ) length scales as follows:

$$L = a(a/h)^{1/2}, \qquad \ell = a \tag{2}$$

While we keep the circumferential length scale as same as the inner radius (a) of the shell, the longitudinal length scale is the longest we can physically describe. Insert the equation (2) into stress displacement and equilibrium equations and use a small thin shell parameter λ = h/a, where h is the total thickness and a is the inner radius of the shell, we will obtain the following equations.
Stress-displacement relations

$$v_{r,y} = \lambda[\bar{S}_{31}t_z + \bar{S}_{32}t_\theta + \bar{S}_{33}t_r + \bar{S}_{34}t_{r\theta} + \bar{S}_{35}t_{rz} + \bar{S}_{36}t_{\theta z}]$$

$$v_{z,y} + \lambda^{3/2}v_{r,x} = \lambda[\bar{S}_{51}t_z + \bar{S}_{52}t_\theta + \bar{S}_{53}t_r + \bar{S}_{54}t_{r\theta} + \bar{S}_{55}t_{rz} + \bar{S}_{56}t_{\theta z}]$$

$$\lambda v_{r,\phi} + (1+\lambda y)v_{\theta,y} - \lambda v_\theta$$

$$= \lambda(1+\lambda y)[\bar{S}_{41}t_z + \bar{S}_{42}t_\theta + \bar{S}_{43}t_r + \bar{S}_{44}t_{r\theta} + \bar{S}_{45}t_{rz} + \bar{S}_{46}t_{\theta z}]$$

$$\lambda^{1/2}v_{z,x} = \bar{S}_{11}t_z + \bar{S}_{12}t_\theta + \bar{S}_{13}t_r + \bar{S}_{14}t_{r\theta} + \bar{S}_{15}t_{rz} + \bar{S}_{16}t_{\theta z}$$

$$v_{\theta,\phi} + v_r = (1+\lambda y)[\bar{S}_{21}t_z + \bar{S}_{22}t_\theta + \bar{S}_{23}t_r + \bar{S}_{24}t_{r\theta} + \bar{S}_{25}t_{rz} + \bar{S}_{26}t_{\theta z}]$$

$$\lambda^{1/2}(1+\lambda y)v_{\theta,x} + v_{z,\phi}$$

$$= (1+\lambda y)[\bar{S}_{61}t_z + \bar{S}_{62}t_\theta + \bar{S}_{63}t_r + \bar{S}_{64}t_{r\theta} + \bar{S}_{65}t_{rz} + \bar{S}_{66}t_{\theta z}] \tag{3}$$

Equilibrium equations

$$[t_{rz}(1+\lambda y)]_{,y} + \lambda t_{\theta z,\phi} + \lambda^{3/2}(1+\lambda y)t_{z,x} = 0$$

$$[t_{r\theta}(1+\lambda y)]_{,y} + \lambda t_{\theta,\phi} + \lambda t_{r\theta} + \lambda^{3/2}(1+\lambda y)t_{\theta z,x} = 0$$

$$[t_r(1+\lambda y)]_{,y} + \lambda t_{r\theta,\phi} + \lambda^{3/2}(1+\lambda y)t_{rz,x} - \lambda t_\theta = 0 \tag{4}$$

By using the asymptotic expansion parameter λ is very small, as shown in the References (12) and (13) we can obtain the desired approximate systems of shell theory.

$$v^{(0)}_{r,y} = 0$$

$$v^{(1)}_{z,y} = 0$$

$$v^{(0)}_{\theta,y} = 0$$

$$v^{(1)}_{z,x} = S^{(0)}_{11} t^{(2)}_{z} + S^{(0)}_{12} t^{(2)}_{\theta}$$

$$v^{(0)}_{\theta,\phi} + v^{(0)}_{r} = 0$$

$$v^{(0)}_{\theta,x} + v^{(1)}_{z,\phi} = 0$$

$$t^{(5)}_{rz,y} + t^{(3)}_{\theta z,\phi} + t^{(2)}_{z,x} = 0$$

$$t^{(4)}_{r\theta,y} + t^{(2)}_{\theta,\phi} = 0$$

$$t^{(4)}_{r,y} - t^{(2)}_{\theta} = 0$$

(5)

The first three equations of above integrated and obtained as follows:

$$v^{(0)}_{r} = V^{(0)}_{r}(x, \phi)$$

$$v^{(1)}_{z} = V^{(1)}_{z}(x, \phi)$$

$$v^{(0)}_{\theta} = V^{(0)}_{\theta}(x, \phi)$$

(6)

Where V_r, V_z and V_θ are the components of the initial displacement at y=0 or r=a surface.

By substituting the equation (6) into next three equations we can obtain the following stress strain equations:

$$V^{(1)}_{z,x} = S^{(0)}_{11} t^{(2)}_{z} + S^{(0)}_{12} t^{(2)}_{\theta}$$

$$V^{(0)}_{\theta,\phi} + V^{(0)}_{r} = 0$$

$$V^{(0)}_{\theta,x} + V^{(1)}_{z,\phi} = 0 \tag{7}$$

The last three equations of the equations (5), (6) and (7) can be examined and interpreted as the characteristics of the following phenomena.

Transverse Strains; the first three equations of (5) are zero and all displacements shown in (6) independent of thickness coordinate, r, which means it is only membrane state.

In-plane Circumferential and Shear Strains; the longitudinal strain is represented by the combination of longitudinal and circumferential stresses (t_z and t_θ)

In-plane Shear Stresses; t_{rz} does not appear in the first approximation theory.

We now complete the second approximation theory of the asymptotic expansion and integration procedure, which can be shown as follows:

$$v^{(2)}_{r,y} = 0$$

$$v^{(3)}_{z,y} + v^{(0)}_{r,x} = 0$$

$$v^{(2)}_{\theta,y} + v^{(0)}_{r,\phi} - v^{(0)}_{\theta} + yv^{(0)}_{\theta,y} = 0$$

$$v^{(3)}_{z,x} = S^{(0)}_{11} t^{(4)}_{z} + S^{(0)}_{12} t^{(4)}_{\theta} + S^{(2)}_{11} t^{(2)}_{z} + S^{(2)}_{12} t^{(2)}_{\theta}$$

$$v^{(2)}_{\theta,\phi} + v^{(2)}_{r} = S^{(0)}_{21} t^{(2)}_{z} + S^{(0)}_{22} t^{(2)}_{\theta}$$

$$v^{(2)}_{\theta,x} + v^{(3)}_{z,\phi} + yv^{(0)}_{\theta,x} = S^{(0)}_{66} t^{(3)}_{\theta z}$$

$$t^{(7)}_{rz,y} + (yt^{(5)}_{rz})_{,y} + t^{(5)}_{\theta Z,\phi} + yt^{(2)}_{z,x} + t^{(4)}_{z,x} = 0$$

$$t^{(6)}_{r\theta,y} + (yt^{(4)}_{r\theta})_{,y} + t^{(4)}_{\theta,\phi} + t^{(4)}_{r\theta} + t^{(3)}_{\theta z,x} = 0$$

$$t^{(6)}_{r,y} + (yt^{(4)}_{r})_{,y} + t^{(4)}_{r\theta,\phi} - t^{(4)}_{\theta} = 0 \tag{8}$$

By integrating the first three equations of the above, we will obtain the following equations:

$$v_r^{(2)} = V_r^{(2)}(x,\phi) \quad , \qquad v_z^{(3)} = V_z^{(3)}(x,\phi) - V_{r,x}^{(0)} y$$
$$v_\theta^{(2)} = V_\theta^{(2)}(x,\phi) + (V_\theta^{(0)} - V_{r,\phi}^{(0)}) y \tag{9}$$

Inserting the displacements obtained in the equation (9) into the middle three equations of (8) to obtain the following equations:

$$V_{z,x}^{(3)} - V_{r,xx}^{(0)} y = S_{11}^{(0)} t_z^{(4)} + S_{12}^{(0)} t_\theta^{(4)} + S_{11}^{(2)} t_z^{(2)} + S_{12}^{(2)} t_\theta^{(2)}$$
$$S_{21}^{(0)} t_z^{(2)} + S_{22}^{(0)} t_\theta^{(2)} = V_{\theta,\phi}^{(2)} + (V_{\theta,\phi}^{(0)} - V_{r,\phi\phi}^{(0)}) y + V_r^{(2)}$$
$$S_{66}^{(0)} t_{\theta z}^{(3)} = V_{\theta,x}^{(2)} + V_{z,\phi}^{(3)} + 2(V_{\theta,x}^{(0)} - V_{r,x\phi}^{(0)}) y \tag{10}$$

By combining the equation for $V_{z,x}$ in the equation (7) and the last two equations of (9), we can obtain the in-plane stress-strain relations as follows:

$$\begin{Bmatrix} t_z^{(2)} \\ t_\theta^{(2)} \\ t_{\theta z}^{(3)} \end{Bmatrix} = [\ C\ ] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} + [\ C\ ] \begin{Bmatrix} K_1 \\ K_2 \\ K_{12} \end{Bmatrix} y \tag{11}$$

where

$$[\ C\ ] = \begin{bmatrix} S_{11}^{(0)} & S_{12}^{(0)} & 0 \\ S_{21}^{(0)} & S_{22}^{(0)} & 0 \\ 0 & 0 & S_{66}^{(0)} \end{bmatrix}^{-1}$$

$$\varepsilon_1 = V_{z,x}^{(1)}$$
$$\varepsilon_2 = V_{\theta,\phi}^{(2)} + V_r^{(2)}$$
$$\varepsilon_{12} = V_{\theta,x}^{(2)} + V_{z,\phi}^{(3)}$$

$$K_1 = 0$$
$$K_2 = V_{\theta,\phi}^{(0)} - V_{r,\phi\phi}^{(0)}$$
$$K_{12} = 2(V_{\theta,x}^{(0)} - V_{r,x\phi}^{(0)}) \tag{12}$$

Substituting t_z, t_θ and $t_{\theta z}$ into the first approximation equations of (5), we will obtain the following transverse stresses:

$$
\begin{aligned}
t_{rz}^{(5)} &= T_{rz}^{(5)}(x,\phi) - [(V_{\theta,x}^{(2)} + V_{z,\phi\phi}^{(3)})A_{33} + 2(V_{\theta,x\phi}^{(0)} - V_{r,x\phi\phi}^{(0)})B_{33}] \\
&\quad -[V_{z,xx}^{(1)} A_{11} + (V_{\theta,x\phi}^{(2)} + V_{r,x}^{(2)})A_{12} + (V_{\theta,x\phi}^{(0)} - V_{r,x\phi\phi}^{(0)})B_{12}] \\
t_{r\theta}^{(4)} &= T_{r\theta}^{(4)}(x,\phi) - [V_{z,x\phi}^{(1)} A_{12} + (V_{\theta,\phi\phi}^{(2)} + V_{r,\phi}^{(2)})A_{22} + (V_{\theta,\phi\phi}^{(0)} - V_{r,\phi\phi\phi}^{(0)})B_{22}] \\
t_{r}^{(4)} &= T_{r}^{(4)}(x,\phi) + [V_{z,x}^{(1)} A_{12} + (V_{\theta,\phi}^{(2)} + V_{r}^{(2)})A_{22} + (V_{\theta,\phi}^{(0)} - V_{r,\phi\phi}^{(0)})B_{12}]
\end{aligned}
\tag{13}
$$

The boundary conditions at the inner and the outer surface of the shell can be specified as follows:

$$
\begin{aligned}
&t_{rz}^{(5)} = t_{r\theta}^{(4)} = t_{r}^{(4)} = 0, && (y=0) \\
&t_{rz}^{(5)} = t_{r\theta}^{(4)} = 0, \qquad t_{r}^{(4)} = p^*, && (y=1)
\end{aligned}
\tag{14}
$$

While the pressure term will be dimensionless as:

$$p^* = p/(\sigma \lambda^2) \tag{15}$$

Note that p* is axi-symmetric and only allowed to vary in longitudinal direction.

$$p^* = p^*(x) \tag{16}$$

After mathematical manipulation and inserting boundary conditions, we can obtain the following final equations:

$$[\underline{D}_{12}-(\underline{D}_{22}\underline{A}_{12}/\underline{A}_{22})]V_{z,x\phi\phi}+[\underline{A}_{11}-(\underline{A}_{12}^{2}/\underline{A}_{22})]V_{z,xxx}+[\underline{D}_{12}-(\underline{D}_{22}\underline{A}_{12}/\underline{A}_{22})]V_{z,x\phi\phi\phi\phi}$$

$$+[\underline{E}_{22}-(\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{\theta,\phi\phi\phi}+[\underline{B}_{12}-(\underline{A}_{12}\underline{B}_{22}/\underline{A}_{22})]V_{\theta,xx\phi}+$$

$$[\underline{E}_{22}-(\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{\theta,\phi\phi\phi\phi\phi}+[-\underline{B}_{12}+(\underline{A}_{12}\underline{B}_{22}/\underline{A}_{22})]V_{r,xx\phi}-$$

$$[\underline{E}_{22}-(\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{r,\phi\phi\phi\phi}-[\underline{E}_{22}-(\underline{B}_{22}\underline{D}_{22}/\underline{A}_{22})]V_{r,\phi\phi\phi\phi\phi\phi}+$$

$$(\underline{A}_{12}/\underline{A}_{22})p^{*}{}_{,xx}=0$$

(17)

where

$$\underline{E}_{ij}=\int_0^y E_{ij}\,d\rho$$

(18)

Finally the above equations are the governing equations of the semi-membrane theory of hybrid anisotropic materials. Note that the higher derivatives are those associated with the circumferential coordinate φ. This is due to the shorter circumferential length scale compare to the longer longitudinal length scale. The equations are further simplified if we consider the physical nature of phenomena, which are axi-symmetric load and axi-symmetric deformations and stress variations.

CONCLUSION

Semi-membrane theory of circular cylindrical shell is developed and formulated by adopting circumferential length scale as inner radius (a) and longitudinal length scale $a(a/h)^{1/2}$ which is extended to hybrid anisotropic materials and the governing equations are well compared to classical theories of Vlasov, Calladine and Gould, etc. which are of isotropic material as shown in the references [6],[18] and [22] respectively..

Because we adopted the circumferential length scale is equal to the inner radius, the developed theory is membrane state circumferentially as shown in the pseudo membrane theory formulated by Chung and Hong as shown in the reference [13]. Longitudinally, a long length scale was adopted, therefore radial deformation and circumferential stresses are of exponentially decaying patterns as explained by Vlasov, Calladine and Vinson as shown in the references [6], [18] and [21] respectively..

Due to the complexity of hybrid anisotropic material being used, the developed theory is of higher order partial differential equations but it should be considerably simplified considering the axi-symmetric nature of load and deformations and very long longitudinal length scale.

The developed theory is very useful for the design of outer space rocket structures as well as open top oil storage tanks.

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REFERENCES

[1] Love,A.E.H, "A treatise on the mathematical theory of elasticity", Dover Edition, New York
[2] Donnell, L.H., "Beams, Plates and Shells", McGraw-Hill Book Company, ISBN0-07-017593
[3] Reissner, E., "The effect of transverse shear deformation on the bending of elastic plates", J. Appl. Mech.12, No A69-77, 1945
[4] Johnson, M.W. and Widera, O.E., "An asymptotic dynamic theory for cylindrical shells", Studies Appl. Math. 48,205, 1969
[5] WIDERA, O. E., "An asymptotic theory for the motion of elastic plates", Acta mechanica 9, 54, 1970
[6] Vlasov, V.Z., "General theory of shells and it's application in engineering", NASA TT F-99, National Tech. Information Service.
[7] Vlasov, V.S., "Basic Differential Equations in General Theory of Elastic Shells", NACA Technical Memorandum 1241, 1951.
[8] Goldenveiser, A.L., "Theory of elastic thin shells", Pergamon Press
[9] Ting, T.C.T., "Anisotropic Elasticity", Oxford Engineering Science Series 45, Oxford University Press
[10] Birman, V., "Extension of Vlasov's Semi-Membrane Theory to Reinforced Composite Shells", J. Appl Mech 59 (2), 464-464.
[11] Reddy, J.N., "Mechanics of Laminated Composite Plates and Shells", 2nd Edition, CRC Press
[12] Chung, S.W. and Park, S.M., "A Shell Theory of Hybrid Anisotropic Materials", International Journal of Composite Materials, Volume 6, Number1, February 2016
[13] Chung, S.W. and Hong, S.G., "Pseudo Membrane Shell Theory of Hybrid Anisotropic Materials", Journal of Composite Structures, Volume 160, Number1, January 2017
[14] ASME Boiler Code - Boiler & Pressure Vessel Code, 2015
[15] ACI 318-11: Building Code Requirements for Structural Concrete and Commentary
[16] Axelred, E.L., "Theory of Flexible Shells", North-Holland Series in Applied Mathematics and Mechanics, ISBN 0-444-87954-4(US)
[17] Blaauwendraad, J and Hoefakker, J.H., "Structural Shell Analysis', Springer, ISBN 978-94-007-6700-3
[18] Calladine, C.R., "Theory of Shell Structures", ISBN 0 521 23835 8, Cambridge University Press
[19] Batdorf et al, S.B., "A simplified Method of Elastic-stability Analysis for Thin Cylindrical Shells", NACA Technical Notes No. 1341 through 1345
[20] Ambartsumian, S.A., "Fragments of the Theory of Anisotropic Shells", World Scientific Publishing Co., ISSN 0218-0235

ISSN 2056-5860

[21] Vinson, J.R., "The Behavior of Shells Composed of Isotropic and Composite Materials", Kluwer Academic Publishers, ISBN 0-7923-2113-8

[22] Gould, P.L., "Analysis of Shells and Plates", ISBN 0-387-96585-8, Springer-Verlag

[23] Chung, S.W. , Hong, S.G., Mendenhall, D.A., "Compare and Contrast Bending Shell Theories of Hybrid Anisotropic Materials", International Journal of Composite Materials, p-ISSN: 2166-479X e-ISSN: 2166-49192017; 7(1): 8-19

LIST OF SYMBOLS

a : Inside Radius of Cylindrical Shell

h : Total Thickness of the Shell Wall

S_i : Radius of Each Layer of Wall (i = 1, 2, 3 --- to the number of layer)

L : Longitudinal Length Scale to be defined, Also Actual Length of the Cylindrical Shell

E_i : Young's Moduli in i Direction

G_{ij} : Shear Moduli in i-j Face

S_{ij} : Compliance Matrix of Materials of Each Layer

r : Radial Coordinate

l : Circumferential Length Scale to be defined

γ : Angle of Fiber Orientation

σ : Normal Stresses

ε : Strains Normal

z, θ, r : Generalized Coordinates in Longitudinal, Circumferential and Radial Directions Respectively

τ : Shear Stresses

ε_{ij} : Shear Strains in i-j Face

λ ; Shell Thickness / Inside Radius (h/a)

C_{ij} : Elastic Moduli in General

X, φ, Y : Non Dimensional Coordinate System in Longitudinal, Circumferential and Radial Directions Respectively

Table 1 List of Symbols

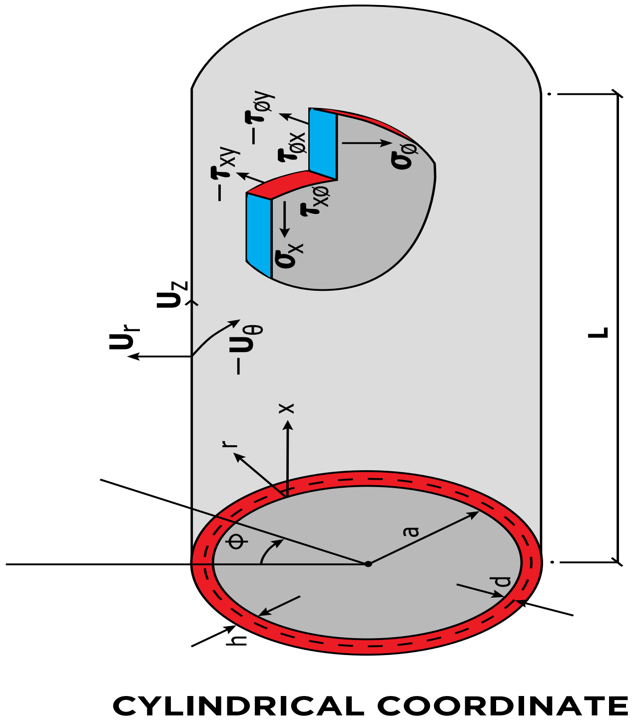

Figure 1. Dimensions, Deformations and Stresses of the Cylindrical Shell

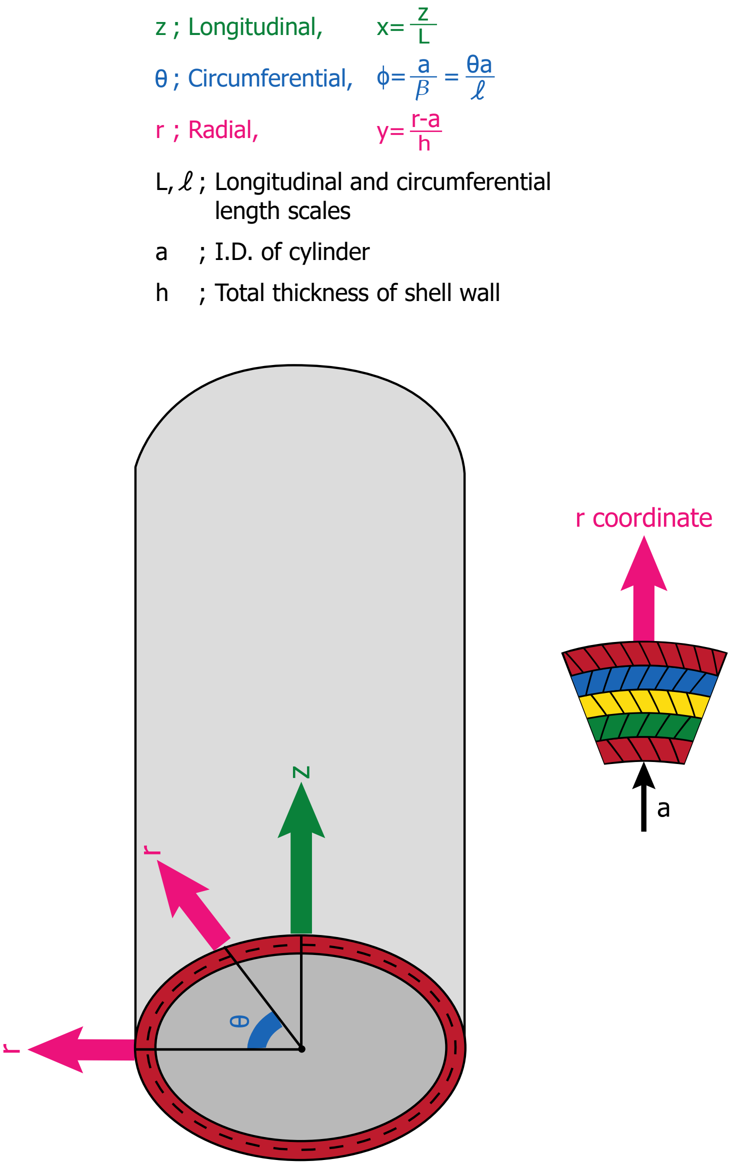

Figure 2, Details of the Coordinate System

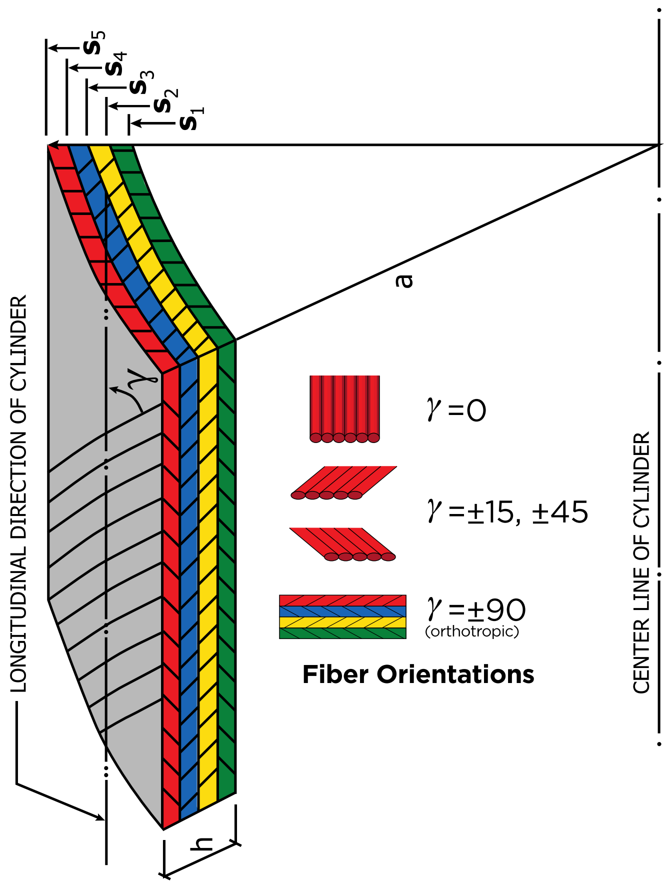

Figure 3, A Laminated Cylindrical Shell, Material Orientation γ

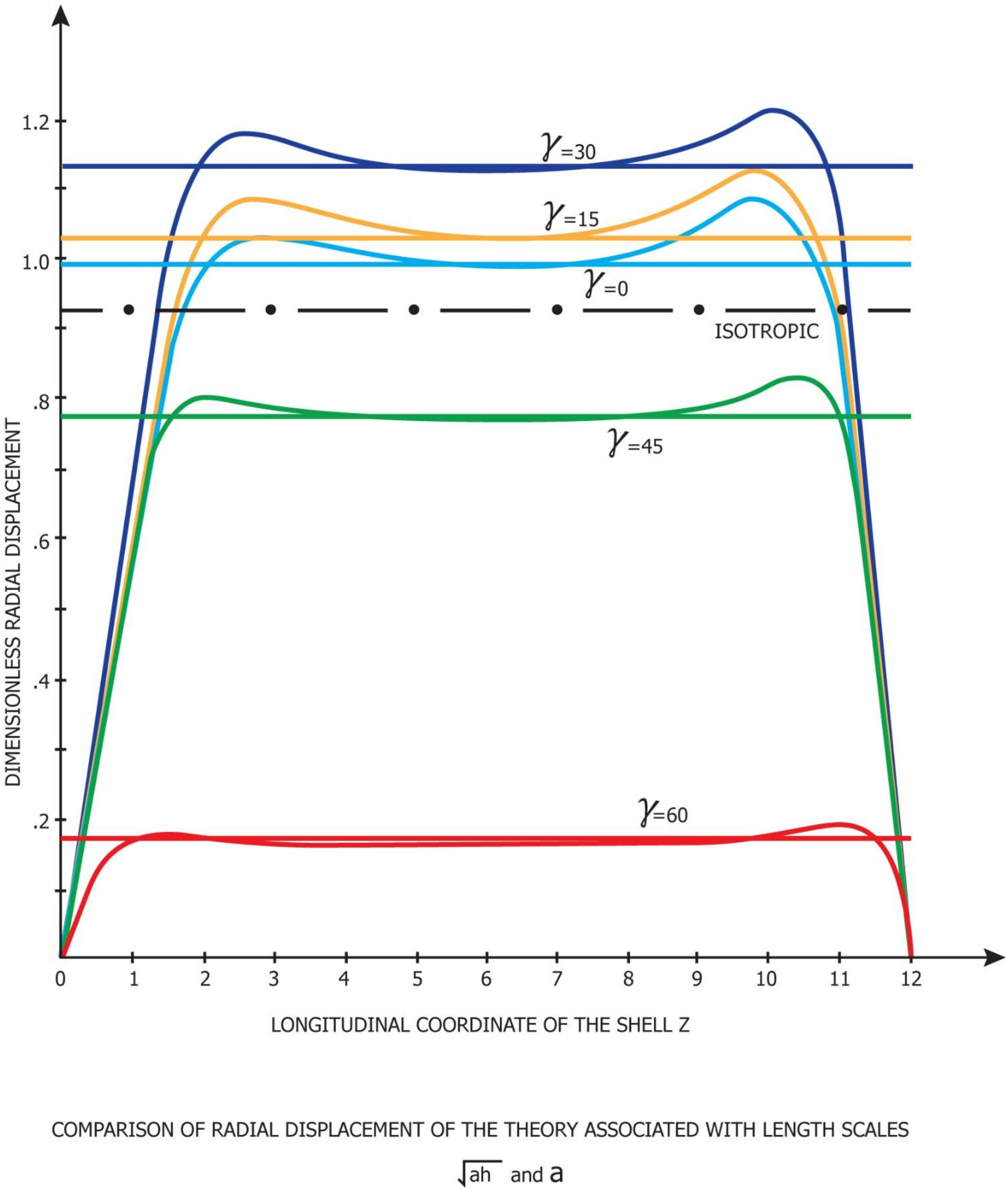

Figure 4, Comparison of Radial Displacements of the Bending and Pure Membrane Theories.