THE EXISTENCE OF SOLUTIONS FOR A CLASS OF IMPULSIVE FRACTIONAL *Q*-DIFFERENCE EQUATIONS

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ABSTRACT

In this paper, we prove the existence and uniqueness of solutions for a class of initial value problem for impulsive fractional q-difference equation of order $1<\alpha\le 2$ by applying some well-known fixed point theorems. Some examples are presented to illustrate the main results.
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INTRODUCTION

In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results on q -difference and fractional q -difference equations can be found in the papers [1-13] and the references cited therein. In [14] the notions of q_k-derivative and q_k -integral of a function $f: J_k := [t_k, t_{k+1}] \to \mathrm{R}$ have been introduced and their basic properties was proved. As applications existence and uniqueness results for initial value problems for first and second order impulsive q_k -difference equations are proved. In [15] , the authors applied the concepts of quantum calculus developed in [14] to study a class of boundary value problem of ordinary impulsive q_k-integro-difference equations, some existence and uniqueness results for this problem were proved by using a variety of fixed point theorems. In [16] the authors used the q -shifting operator to develop the new concepts of fractional quantum calculus such as the Riemann–Liouville fractional derivative and integral and their properties. They also formulated the existence and uniqueness results for some classes of first and second orders impulsive fractional q -difference equations. Inspired by[16], in this paper, we study the existence and uniqueness of solutions for the following initial value problem for impulsive fractional q-differ- ence equation of order $1<\alpha\le 2$ the form

$$\begin{cases} {}_{t_k}D_{q_k}^{\alpha} x(t) = f(t, x(t)),\ t\in J, t\neq t_k \\ \Delta x(t_k) = \varphi_k(x(t_k)),\ k=1,2,\ldots,m, \\ \Delta^* x(t_k) = \varphi_k^*(x(t_k)), k=1,2,\ldots,m, \\ x(0)=0,\ {}_0D_{q_0}^{\alpha-1}x(0) = \beta\, {}_{t_{k_0}}D_{q_{k_0}}^{\alpha-1} x(\eta), \end{cases} \tag{1.1}$$

where $J=[0,T]$, $0=t_0<t_1<t_2<\cdots<t_k<\cdots<t_m<t_{m+1}=T$, $J_0=[t_0,t_1]$, $J_k=(t_k,t_{k+1}]$, $k=1,2,\ldots,m$. ${}_{t_k}D_{q_k}^{\alpha}$ and ${}_{t_k}D_{q_k}^{\alpha-1}$ respectively are the Riemann-Liouville fractional q-difference of order α and $\alpha-1$ on interval J_k, $0<q_k<1$ for $k=1,2,\ldots,m$, $f: J\times\mathrm{R}\to\mathrm{R}$ is a continuous function, $\varphi_k, \varphi_k^* \in C(\mathrm{R},\mathrm{R})$ for $k=1,2,\ldots,m$. The notation $\Delta x(t_k)$ and $\Delta^* x(t_k)$ are defined by

$$\begin{aligned} \Delta x(t_k) &= {}_{t_k}I_{q_k}^{1-\alpha} x(t_k^+) - {}_{t_{k-1}}I_{q_{k-1}}^{1-\alpha} x(t_k),\ k=1,2,\ldots,m, \\ \Delta^* x(t_k) &= {}_{t_k}I_{q_k}^{2-\alpha} x(t_k^+) - {}_{t_{k-1}}I_{q_{k-1}}^{2-\alpha} x(t_k), k=1,2,\ldots,m, \end{aligned} \tag{1.2}$$

where ${}_{t_k}I_{q_k}^{1-\alpha}$ and ${}_{t_k}I_{q_k}^{2-\alpha}$ respectively are the Riemann-Liouville fractional q-integral of order $1-\alpha$ and $2-\alpha$ on J_k. $\beta\in\mathrm{R}$, $k_0\in\{1,2,\cdots,m\}$, $\eta\in(t_{k_0}, t_{k_0+1}]$.

Preliminaries

This section is devoted to some basic concepts such as q-shifting operator, Riemann–Liouville fractional q-integral and q-difference on a given interval. The presentation here can be found in, for example, [16,17].

We define a q-shifting operator as

$$_{a}\Phi_{q}(m)=qm+(1-q)a\,.$$

The power of q-shifting operator is defined as

$$_{a}(n-m)_{q}^{(0)}=1,\ _{a}(n-m)_{q}^{(k)}=\prod_{i=0}^{k-1}(n-{}_{a}\Phi_{q}^{i}(m)),\ k\in \mathrm{N}\cup\{\infty\}\,,$$

More generally, if $\gamma\in \mathrm{R}$, then

$$_{a}(n-m)_{q}^{(\gamma)}=n^{(\gamma)}\prod_{i=0}^{\infty}\frac{1-{}_{a/n}\Phi_{q}^{i}(m/n)}{1-{}_{a/n}\Phi_{q}^{\gamma+i}(m/n)}\,.$$

Definition 2.1. The fractional q-derivative of Riemann–Liouville type of order $v\geq 0$ on interval $[a,b]$ is defined by $({}_{a}D_{q}^{0}f)(t)=f(t)$ and

$$({}_{a}D_{q}^{v}f)(t)=({}_{a}D_{qa}^{l}I_{q}^{l-v}f)(t),\ v>0\,,$$

where l is the smallest integer greater than or equal to v.

Definition 2.2. Let $\alpha\geq 0$ and f be a function defined on $[a,b]$. The fractional q-integral of Riemann–Liouville type is given by $({}_{a}I_{q}^{0}f)(t)=f(t)$ and

$$({}_{a}I_{q}^{\alpha}f)(t)=\frac{1}{\Gamma_{q}(\alpha)}\int_{a}^{t}(t-{}_{a}\Phi_{q}(s))_{a}^{(\alpha-1)}f(s)\,{}_{a}d_{q}s,\ \alpha>0,t\in[a,b].$$

From [16] , we have the following formulas for $t\in[a,b],\alpha>0,\beta\in \mathrm{R}$:

$$_{a}D_{q}^{\alpha}(t-a)^{\beta}=\frac{\Gamma_{q}(\beta+1)}{\Gamma_{q}(\beta-\alpha+1)}(t-a)^{\beta-\alpha},\quad {}_{a}I_{q}^{\alpha}(t-a)^{\beta}=\frac{\Gamma_{q}(\beta+1)}{\Gamma_{q}(\beta+\alpha+1)}(t-a)^{\beta+\alpha}\,.$$

Lemma 2.3. *Let $\alpha,\beta\in \mathrm{R}^{+}$ and f be a continuous function on $[a,b],a\geq 0$. The Riemann–Liouville fractional q-integral has the following semi-group property*

$$_{a}I_{q}^{\beta}{}_{a}I_{q}^{\alpha}f(t)={}_{a}I_{q}^{\alpha}{}_{a}I_{q}^{\beta}f(t)={}_{a}I_{q}^{\alpha+\beta}f(t)\,.$$

Lemma 2.4. *Let f be a q-integrable function on $[a,b]$. Then the following equality holds*

$$_{a}D_{q}^{\alpha}{}_{a}I_{q}^{\alpha}f(t)=f(t)\,.\ \text{For}\ \alpha>0,t\in[a,b]\,.$$

Lemma 2.5. *Le t $\alpha>0$ and p be a positive integer. Then for $t\in[a,b]$ the following equality holds*

$$_{a}I_{q}^{\alpha}{}_{a}D_{q}^{p}f(t)={}_{a}D_{q}^{p}{}_{a}I_{q}^{\alpha}f(t)-\sum_{k=0}^{p-1}\frac{(t-a)^{\alpha-p+k}}{\Gamma_{q}(\alpha+k-p+1)}{}_{a}D_{q}^{k}f(a)\,.$$

Lemma 2.6. ([18])*Let E be a Banach space. Assume that Ω is an open bounded subset of E with $\theta\in\Omega$ and let $T:\overline{\Omega}\to E$ be a completely continuous operator such that*

$$\|Tu\|\leq\|u\|,\forall u\in\partial\Omega\,.$$

Then T has a fixed point in $\overline{\Omega}$.

Lemma 2.7. ([18]). *Let E be a Banach space. Assume that $T:E\to E$ is a completely continuous operator and the set $V=\{u\in E|u=\mu Tu,0<\mu<1\}$ is bounded. Then T has a fixed point in E.*

Let $PC(J,\mathrm{R})=\{x:J\to \mathrm{R}:x(t)$ is continuous everywhere except for some t_k at which $x(t_k^{+})$ and $x(t_k^{-})$

exist and $x(t_k^-)=x(t_k), k=1,2,3,\dots,m\}$. For $\gamma\in\mathrm{R}^+$, we introduce the space $C_{\gamma,k}(J_k,\mathrm{R})=\{x:J_k\to\mathrm{R}:(t-t_k)^\gamma\ \ x(t)\in C(J_k,\mathrm{R})\}$ with the norm $\|x\|_{C_{\gamma,k}}=\sup_{t\in J_k}\left|(t-t_k)^\gamma x(t)\right|$ and $PC_\gamma(J,\mathrm{R})=\{x:J\to\mathrm{R}:$ for each $t\in J_k$ and
$(t-t_k)^\gamma x(t)\in C(J_k,\mathrm{R}), k=0,1,2\dots,m\}$ with the norm $\|x\|_{PC_\gamma}=\max\{\sup_{t\in J_k}\left|(t-t_k)^\gamma x(t)\right|:k=0,1,2\dots,m\}$.
Clearly $PC_\gamma(J,\mathrm{R})$ is a Banach space.

Lemma2.8. *If* $x\in PC(J,\mathrm{R})$ *is a solution of* (1.1), *then for any* $t\in J_k, k=1,2,\dots,m$,

$$x(t)=\frac{m_1(t-t_k)^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}+\frac{m_2(t-t_k)^{\alpha-1}}{\Gamma_{q_k}(\alpha)}+{}_{t_k}I_{q_k}^\alpha f(t,x(t)), \tag{2.1}$$

Where

$$m_1=\frac{\beta t_k}{1-\beta}[{}_{t_{k_0}}I_{q_{k_0}}^1 f(s,x(s))(\eta)+\sum_{0<t_j<\eta}\left({}_{t_{j-1}}I_{q_{j-1}}^1 f(s,x(s))(t_j)+\varphi_j(x(t_j))\right)]$$
$$+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})\left({}_{t_{j-1}}I_{q_{j-1}}^1 f(s,x(s))(t_j)+\varphi_j(x(t_j))\right)+\sum_{0<t_k<t}\left({}_{t_{k-1}}I_{q_{k-1}}^2 f(s,x(s))(t_k)+\varphi_k^*(x(t_k))\right)$$

(2.2)

$$m_2=\frac{\beta}{1-\beta}[{}_{t_{k_0}}I_{q_{k_0}}^1 f(s,x(s))(\eta)+\sum_{0<t_j<\eta}\left({}_{t_{j-1}}I_{q_{j-1}}^1 f(s,x(s))(t_j)+\varphi_j(x(t_j))\right)]$$
$$+\sum_{0<t_k<t}\left({}_{t_{k-1}}I_{q_{k-1}}^1 f(s,x(s))(t_k)+\varphi_k(x(t_k))\right), \tag{2.3}$$

With $\sum_{0<0}(\cdot)=0$.

Proof. For $t\in J_0$, taking the Riemann-Liouville fractional q_0-integral of order α for the first
equation of (1.1) and using Definition 2.1 with Lemma 2.5, we get

$$x(t)=\frac{t^{\alpha-2}}{\Gamma_{q_0}(\alpha-1)}C_0+\frac{t^{\alpha-1}}{\Gamma_{q_0}(\alpha)}C_1+{}_0I_{q_0}^\alpha f(t,x(t)) \tag{2.4}$$

where $C_0={}_0I_{q_0}^{2-\alpha}x(0)$ and $C_1={}_0I_{q_0}^{1-\alpha}x(0)$.The first initial condition of (1.1) implies that C_0=0 .Taking the Riemann-Liouville fractional q_0-derivative of order $\alpha-1$ for (2.4) on J_0, we have

$${}_0D_{q_0}^{\alpha-1}x(t)=C_1+{}_0I_{q_0}^1 f(t,x(t)),$$

And ${}_0D_{q_0}^{\alpha-1}x(0)=C_1$.Therefore, (2.4) can be written as

$$x(t)=\frac{t^{\alpha-1}}{\Gamma_{q_0}(\alpha)}C_1+{}_0I_{q_0}^\alpha f(t,x(t)). \tag{2.5}$$

Applying the Riemann-Liouville fractional q_0-derivative of orders $1-\alpha$ and $2-\alpha$ for (2.5) at t=t_1, we have

$${}_0I_{q_0}^{1-\alpha}x(t_1)=C_1+{}_0I_{q_0}^1 f(s,x(s))(t_1),\quad {}_0I_{q_0}^{2-\alpha}x(t_1)=C_1t_1+{}_0I_{q_0}^2 f(s,x(s))(t_1), \tag{2.6}$$

For $t\in J_1$=$(t_1,t_2]$, Riemann-Liouville fractional q_1-integrating (1.1), we obtain

$$x(t)=\frac{(t-t_1)^{\alpha-2}}{\Gamma_{q_1}(\alpha-1)}{}_{t_1}I_{q_1}^{2-\alpha}x(t_1^+)+\frac{(t-t_1)^{\alpha-1}}{\Gamma_{q_1}(\alpha)}{}_{t_1}I_{q_1}^{1-\alpha}x(t_1^+)+{}_{t_1}I_{q_1}^\alpha f(t,x(t)), \tag{2.7}$$

Using the jump conditions of equation (1.1) with (2.6)-(2.7) for $t\in J_1$, we get

$$x(t)=\frac{(t-t_1)^{\alpha-2}}{\Gamma_{q_1}(\alpha-1)}[C_1t_1+{}_0I_{q_0}^2 f(s,x(s))(t_1)+\varphi_1^*(x(t_1)]+\frac{(t-t_1)^{\alpha-1}}{\Gamma_{q_1}(\alpha)}[C_1+{}_0I_{q_0}^1 f(s,x(s))(t_1)+\varphi_1(x(t_1)]+{}_{t_1}I_{q_1}^\alpha f(t,x(t))$$

.

Repeating the above process, for $t\in J_k$=$(t_k,t_{k+1}]$, we obtain

$$x(t)=\frac{(t-t_k)^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}[C_1t_k+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})\left({}_{t_{j-1}}I^1_{q_{j-1}}f(s,x(s))(t_j)+\varphi_j(x(t_j))\right)$$
$$+\sum_{0<t_k<t}\left({}_{t_{k-1}}I^2_{q_{k-1}}f(s,x(s))(t_k)+\varphi^*_k(x(t_k))\right)]+\frac{(t-t_k)^{\alpha-1}}{\Gamma_{q_k}(\alpha)}[C_1 \tag{2.8}$$
$$+\sum_{0<t_k<t}\left({}_{t_{k-1}}I^1_{q_{k-1}}f(s,x(s))(t_k)+\varphi_k(x(t_k))\right)]+{}_{t_k}I^\alpha_{q_k}f(t,x(t)),$$

Taking the Riemann-Liouville fractional q_k-derivative of order $\alpha-1$ for (2.8) and using $\Gamma_{q_k}(0)=\infty$,

it follows that

$${}_{t_k}D^{\alpha-1}_{q_k}x(t)=C_1+\sum_{0<t_j<t}\left({}_{t_{j-1}}I^1_{q_{j-1}}f(s,x(s))(t_j)+\varphi_j(x(t_j))\right)+{}_{t_k}I^1_{q_k}f(t,x(t)).$$

For $k_0\in\{1,2,\cdots,m\}$, $\eta\in(t_{k_0},t_{k_0+1}]$, we have

$${}_{t_{k_0}}D^{\alpha-1}_{q_{k_0}}x(\eta)=C_1+\sum_{0<t_j<\eta}\left({}_{t_{j-1}}I^1_{q_{j-1}}f(s,x(s))(t_j)+\varphi_j(x(t_j))\right)+{}_{t_{k_0}}I^1_{q_{k_0}}f(s,x(s))(\eta).$$

The initial condition ${}_0D^{\alpha-1}_{q_0}x(0)=\beta_{t_{k_0}}D^{\alpha-1}_{q_{k_0}}x(\eta)$ leads to

$$C_1=\frac{\beta}{1-\beta}[\sum_{0<t_j<\eta}\left({}_{t_{j-1}}I^1_{q_{j-1}}f(s,x(s))(t_j)+\varphi_j(x(t_j))\right)+{}_{t_{k_0}}I^1_{q_{k_0}}f(s,x(s))(\eta)].$$

Substituting the value of C_1 in (2.8), we obtain (2.1). Conversely, assume that x is a solution of the impulsive fractional integral equation (2.1), then by a direct computation, it follows that the solution given by(2.1)satisfies equation (1.1). This completes the proof.

Main results

This section deals with the existence and uniqueness of solutions for the equation (1.1) . In view of Lemma 2.8 , we define an operator $A:PC(J,\mathrm{R})\to PC(J,\mathrm{R})$ by

$$(Ax)(t)=\frac{m_1(t-t_k)^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}+\frac{m_2(t-t_k)^{\alpha-1}}{\Gamma_{q_k}(\alpha)}+{}_{t_k}I^\alpha_{q_k}f(t,x(t)),$$

where m_1, m_2 are given by (2.2) and (2.3).

Theorem 3.1. *Let* $\lim_{x\to0}\frac{f(t,x)}{x}=0,\lim_{x\to0}\frac{\varphi_k(x)}{x}=0$ *and* $\lim_{x\to0}\frac{\varphi^*_k(x)}{x}=0$ $(k=1,2,\ldots,m)$, *then equation* (1.1) *has at least one solution.*

Proof. To show that $Ax\in PC_\gamma(J,\mathrm{R})$ for $x\in PC_\gamma(J,\mathrm{R})$, we suppose $\tau_1,\tau_2\in J_k$, and $\tau_1>\tau_2$, then

$$\left|(\tau_1-t_k)^\gamma Ax(\tau_1)-(\tau_2-t_k)^\gamma Ax(\tau_2)\right|$$
$$=\left|(\tau_1-t_k)^\gamma[\frac{m_1(\tau_1-t_k)^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}+\frac{m_2(\tau_1-t_k)^{\alpha-1}}{\Gamma_{q_k}(\alpha)}+{}_{t_k}I^\alpha_{q_k}f(s,x(s))(\tau_1)]\right.$$

$$-(\tau_2-t_k)^{\gamma}[\frac{m_1(\tau_2-t_k)^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}+\frac{m_2(\tau_2-t_k)^{\alpha-1}}{\Gamma_{q_k}(\alpha)}+{}_{t_k}I^{\alpha}_{q_k}f(s,x(s))(\tau_2)]\Big|$$

$$\le\left|\frac{(\tau_1-t_k)^{\gamma+\alpha-2}-(\tau_2-t_k)^{\gamma+\alpha-2}}{\Gamma_{q_k}(\alpha-1)}\right|\{\left|\frac{\beta t_k}{1-\beta}\right|[{}_{t_{k_0}}I^1_{q_{k_0}}|f(s,x(s))|(\eta)+\sum_{0<t_j<\eta}\left({}_{t_{j-1}}I^1_{q_{j-1}}|f(s,x(s))|(t_j)+|\varphi_j(x(t_j))|\right)]$$

$$+\sum_{0<t_k<t}\sum_{0<t_j<t_k}(t_k-t_{k-1})\left({}_{t_{j-1}}I^1_{q_{j-1}}|f(s,x(s)|(t_j))+|\varphi_j(x(t_j))|\right)+\sum_{0<t_k<t}\left({}_{t_{k-1}}I^2_{q_{k-1}}|f(s,x(s))|(t_k)+|\varphi^*_k(x(t_k))|\right)\}$$

$$+\left|\frac{(\tau_1-t_k)^{\gamma+\alpha-1}-(\tau_2-t_k)^{\gamma+\alpha-1}}{\Gamma_{q_k}(\alpha)}\right|\{\left|\frac{\beta}{1-\beta}\right|[{}_{t_{k_0}}I^1_{q_{k_0}}|f(s,x(s))|(\eta)+\sum_{0<t_j<\eta}\left({}_{t_{j-1}}I^1_{q_{j-1}}|f(s,x(s))|(t_j)+|\varphi_j(x(t_j))|\right)]$$

$$+\sum_{0<t_k<t}\left({}_{t_{k-1}}I^1_{q_{k-1}}|f(s,x(s))|(t_k)+|\varphi_k(x(t_k))|\right)\}+\frac{1}{\Gamma_{q_k}(\alpha)}\left|(\tau_1-t_k)^{\gamma}\int_{\tau_2}^{\tau_1}(\tau_1-{}_{t_k}\Phi_{q_k}(s))^{(\alpha-1)}_{t_k}f(s,x(s))\,{}_{t_k}d_{q_k}s\right|$$

$$+\frac{1}{\Gamma_{q_k}(\alpha)}\left|\int_{t_k}^{\tau_2}[(\tau_1-t_k)^{\gamma}(\tau_1-{}_{t_k}\Phi_{q_k}(s))^{(\alpha-1)}_{t_k}-(\tau_2-t_k)^{\gamma}(\tau_2-{}_{t_k}\Phi_{q_k}(s))^{(\alpha-1)}_{t_k}]f(s,x(s))\,{}_{t_k}d_{q_k}s\right|.$$

As $\tau_1\to\tau_2$, we have $|(\tau_1-t_k)^{\gamma}Ax(\tau_1)-(\tau_2-t_k)^{\gamma}Ax(\tau_2)|\to 0$ for each $k=0,1,2,\dots,m$. Therefore, we get $Ax\in PC_{\gamma}(J,\mathrm{R})$. Now we show that the operator $A:PC_{\gamma}(J,\mathrm{R})\to PC_{\gamma}(J,\mathrm{R})$ is completely continuous. Note that A is continuous in view of continuity of f,φ and φ^*. Let $B\subset PC_{\gamma}(J,\mathrm{R})$ be bounded. Then, there exist positive constants $L_i>0$ $(i=1,2,3)$ such that $|f(t,x)|\le L_1$, $|\varphi_k(x)|\le L_2$, $|\varphi^*_k(x)|\le L_3$, $\forall x\in B$. Thus, $\forall x\in B$,
We have

$$|m_1|\le\left|\frac{\beta t_k}{1-\beta}\right|[L_1\,{}_{t_{k_0}}I^1_{q_{k_0}}1(\eta)+\sum_{j=1}^{k_0}{}_{t_{j-1}}I^1_{q_{j-1}}|f(s,x(s)|(t_j)+\sum_{j=1}^{k_0}|\varphi_j(x(t_j))|]$$

$$+\sum_{i=2}^{k}\sum_{j=1}^{i-1}(t_i-t_{i-1})({}_{t_{j-1}}I^1_{q_{j-1}}|f(s,x(s)|(t_j)+|\varphi_j(x(t_j))|)+\sum_{i=1}^{k}({}_{t_{i-1}}I^2_{q_{i-1}}|f(s,x(s)|(t_i)+|\varphi^*_i(x(t_i))|)$$

$$\le\left|\frac{\beta T}{1-\beta}\right|(L_1\eta+k_0L_2)+L_1\sum_{i=1}^{k}(t_i-t_{i-1})t_{i-1}+L_2\sum_{i=1}^{k}(t_i-t_{i-1})(i-1)+L_1\sum_{i=1}^{k}\frac{(t_i-t_{i-1})^2}{1+q_{i-1}}+kL_3,$$

$$|m_2|\le\left|\frac{\beta}{1-\beta}\right|(L_1\eta+k_0L_2)+(L_1t_k+kL_2),$$

$${}_{t_k}I^{\alpha}_{q_k}|f(t,x(t))|\le\frac{L_1(t-t_k)^{\alpha}}{\Gamma_{q_k}(\alpha+1)}.$$

Therefore,

$$\begin{aligned}(t-t_k)^{\gamma}|(Ax)(t)|\le&\frac{(t-t_k)^{\gamma+\alpha-2}}{\Gamma_{q_k}(\alpha-1)}[\left|\frac{\beta T}{1-\beta}\right|(L_1\eta+k_0L_2)+L_1\sum_{i=1}^{k}(t_i-t_{i-1})t_{i-1}\\&+L_2\sum_{i=1}^{k}(t_i-t_{i-1})(i-1)+L_1\sum_{i=1}^{k}\frac{(t_i-t_{i-1})^2}{1+q_{i-1}}+kL_3]\\&+\frac{(t-t_k)^{\gamma+\alpha-1}}{\Gamma_{q_k}(\alpha)}[\left|\frac{\beta}{1-\beta}\right|(L_1\eta+k_0L_2)+L_1t_k+kL_2]+\frac{L_1(t-t_k)^{\alpha+\gamma}}{\Gamma_{q_k}(\alpha+1)}\qquad(3.1)\\\le&\frac{T^{\gamma+\alpha-2}}{\Gamma_{q_k}(\alpha-1)}[\left|\frac{\beta T}{1-\beta}\right|(L_1\eta+mL_2)+mL_3+2L_1mT^2+\frac{m^2TL_2}{2}]\\&+\frac{T^{\gamma+\alpha-1}}{\Gamma_{q_k}(\alpha)}[\left|\frac{\beta}{1-\beta}\right|(L_1\eta+mL_2)+L_1T+mL_2]+\frac{L_1T^{\alpha+\gamma}}{\Gamma_{q_k}(\alpha+1)},\end{aligned}$$

which implies that

$$\|(Ax)(t)\| \le \frac{T^{\gamma+\alpha-2}}{\Gamma_{q_k}(\alpha-1)}[\left|\frac{\beta T}{1-\beta}\right|(L_1\eta+mL_2)+mL_3+2L_1mT^2+\frac{m^2TL_2}{2}]$$
$$+\frac{T^{\gamma+\alpha-1}}{\Gamma_{q_k}(\alpha)}[\left|\frac{\beta}{1-\beta}\right|(L_1\eta+mL_2)+L_1T+mL_2]+\frac{L_1T^{\alpha+\gamma}}{\Gamma_{q_k}(\alpha+1)}:=L.$$

On the other hand, for any $t_1,t_2\in J_k$, with $t_1<t_2, 0\le k\le m$, we have

$$\left|(t_2-t_k)^\gamma(Ax)(t_2)-(t_1-t_k)^\gamma(Ax)(t_1)\right|$$
$$\le \frac{\left|(t_2-t_k)^{\gamma+\alpha-2}-(t_1-t_k)^{\gamma+\alpha-2}\right|}{\Gamma_{q_k}(\alpha-1)}[\left|\frac{\beta T}{1-\beta}\right|(L_1\eta+k_0L_2)+kL_3+2L_1kT^2+\frac{k^2TL_2}{2}]$$
$$+\frac{\left|(t_2-t_k)^{\gamma+\alpha-1}-(t_1-t_k)^{\gamma+\alpha-1}\right|}{\Gamma_{q_k}(\alpha)}[\left|\frac{\beta}{1-\beta}\right|(L_1\eta+k_0L_2)+L_1T+kL_2]$$
$$+\left|(t_2-t_k)^\gamma\,{}_{t_k}I_{q_k}^\alpha f(s,x(s))(t_2)-(t_1-t_k)^\gamma\,{}_{t_k}I_{q_k}^\alpha f(s,x(s))(t_1)\right|\to 0\quad (t_1\to t_2),$$

This implies that A is equicontinuous on all the subintervals $J_k, k=0,1,2,\ldots,m$. Thus, by Arzela–Ascoli Theorem, it follows that $A:PC_\gamma(J,\mathrm{R})\to PC_\gamma(J,\mathrm{R})$ is completely continuous.

Now, in view of $\lim\limits_{x\to 0}\frac{f(t,x)}{x}=0, \lim\limits_{x\to 0}\frac{\varphi_k(x)}{x}=0$ *and* $\lim\limits_{x\to 0}\frac{\varphi_k^*(x)}{x}=0\ (k=1,2,\ldots,m)$, there exists a constant $r>0$ such that $|f(t,x)|\le\delta_1|x|,\ |\varphi_k(x)|\le\delta_2|x|,\ |\varphi_k^*(x)|\le\delta_3|x|$, for $0<|x|<r$, where $\delta_i>0\ (i=1,2,3)$ satisfy

$$\frac{T^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}[\left|\frac{\beta T}{1-\beta}\right|(\delta_1\eta+m\delta_2)+m\delta_3+2\delta_1mT^2+\frac{m^2T\delta_2}{2}]$$
$$+\frac{T^{\alpha-1}}{\Gamma_{q_k}(\alpha)}[\left|\frac{\beta}{1-\beta}\right|(\delta_1\eta+m\delta_2)+\delta_1T+m\delta_2]+\frac{\delta_1T^\alpha}{\Gamma_{q_k}(\alpha+1)}\le 1.$$

Define $\Omega=\{x\in PC_\gamma(J,\mathrm{R}):\|x\|<r\}$ and take $x\in PC_\gamma(J,\mathrm{R})$ such that $\|x\|=r$ so that $x\in\partial\Omega$. Then, by the process used to obtain (3.1), we have

$$(t-t_k)^\gamma|(Ax)(t)|\le\{\frac{T^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}[\left|\frac{\beta T}{1-\beta}\right|(\delta_1\eta+m\delta_2)+m\delta_3+2\delta_1mT^2+\frac{m^2T\delta_2}{2}]$$
$$+\frac{T^{\alpha-1}}{\Gamma_{q_k}(\alpha)}[\left|\frac{\beta}{1-\beta}\right|(\delta_1\eta+m\delta_2)+\delta_1T+m\delta_2]+\frac{\delta_1T^\alpha}{\Gamma_{q_k}(\alpha+1)}\}\|x\|\le\|x\|,$$

which implies that $\|(Ax)(t)\|\le\|x\|, x\in\partial\Omega$.

Therefore, by Lemma 2.6, the operator A has at least one fixed point, which in turn implies that (1.1) has at least one solution $x\in\overline{\Omega}$. This completes the proof.

Theorem 3.2. *Assume that*

(H_1) *there exist positive constants* L_i (i = 1, 2, 3) *such that*

$|f(t,x)|\le L_1,\ |\varphi_k(x)|\le L_2,\ |\varphi_k^*(x)|\le L_3$ for $t\in J, x\in\mathrm{R}$ and $k=1,2,\ldots,m$.

Then equation (1.1) *has at least one solution.*

Proof. As shown in Theorem 3.1, the operator $A:PC_\gamma(J,\mathrm{R})\to PC_\gamma(J,\mathrm{R})$ is completely continuous. Now, we show the set $V=\{x\in PC_\gamma(J,\mathrm{R})|x=\mu Ax, 0<\mu<1\}$ is bounded.

Let $x\in V$, then $x=\mu Ax, 0<\mu<1$. For any $t\in J$, we have

$$x(t)=\frac{\mu m_1(t-t_k)^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}+\frac{\mu m_2(t-t_k)^{\alpha-1}}{\Gamma_{q_k}(\alpha)}+\mu\,{}_{t_k}I_{q_k}^\alpha f(t,x(t)),\qquad (3.2)$$

where m_1, m_2 are given by (2.2) and (2.3). Combining (H_1) and (3.2), we obtain

$$(t-t_k)^{\gamma}|x(t)| \le \frac{\mu|m_1|(t-t_k)^{\gamma+\alpha-2}}{\Gamma_{q_k}(\alpha-1)} + \frac{\mu|m_2|(t-t_k)^{\gamma+\alpha-1}}{\Gamma_{q_k}(\alpha)} + \mu(t-t_k)^{\gamma}\, {}_{t_k}I_{q_k}^{\alpha}|f(t,x(t))|$$

$$\le \frac{\mu T^{\gamma+\alpha-2}}{\Gamma_{q_k}(\alpha-1)}\left[\left|\frac{\beta T}{1-\beta}\right|(L_1\eta+mL_2)+mL_3+2L_1mT^2+\frac{m^2TL_2}{2}\right]$$

$$+\frac{\mu T^{\gamma+\alpha-1}}{\Gamma_{q_k}(\alpha)}\left[\left|\frac{\beta}{1-\beta}\right|(L_1\eta+mL_2)+L_1T+mL_2\right]+\frac{\mu L_1 T^{\alpha+\gamma}}{\Gamma_{q_k}(\alpha+1)} := L,$$

Thus, for any $t \in J$, it follows that $\|x\| \le L$. So, the set V is bounded. Therefore, by the conclusion of Lemma 2.7, the operator A has at least one fixed point. This impliesthat (1.1) has at least one solution. This completes the proof.

Theorem 3.3. *Assume that*

(H_2) *there exist positive constants N_i (i = 1, 2, 3) such that*

$$|f(t,x)-f(t,y)| \le N_1|x-y|,\ |\varphi_k(x)-\varphi_k(y)| \le N_2|x-y|,\ |\varphi_k^*(x)-\varphi_k^*(y)| \le N_3|x-y|$$

for $t \in J, x \in \mathrm{R}$ *and* $k = 1,2,\ldots,m$.

Then equation (1.1) *has a unique solution if*

$$\Lambda = \frac{T^*}{\Gamma^*}\left[\left|\frac{\beta}{1-\beta}\right|(N_1\eta+mN_2)(T+1)+mN_3+N_1(1+T+2mT^2)+N_2(m+\frac{m^2T}{2})\right] < 1, \qquad (3.3)$$

Where $T^* = \max\{T^{\alpha-2}, T^{\alpha-1}, T^{\alpha}\}$ $\Gamma^* = \min\{\Gamma_{q_k}(\alpha-1), \Gamma_{q_k}(\alpha), \Gamma_{q_k}(\alpha+1)\}$.

Proof. For $x, y \in PC_{\gamma}(J,\mathrm{R})$, we have

$$(t-t_k)^{\gamma}|(Ax)(t)-(Ay)(t)| \le \frac{(t-t_k)^{\gamma+\alpha-2}}{\Gamma_{q_k}(\alpha-1)}\{\left|\frac{\beta T}{1-\beta}\right|[{}_{t_{k_0}}I_{q_{k_0}}^{1}|f(s,x(s))-f(s,y(s))|(\eta)$$

$$+\sum_{j=1}^{k_0} {}_{t_{j-1}}I_{q_{j-1}}^{1}|f(s,x(s)-f(s,y(s)|(t_j)+\sum_{j=1}^{k_0}|\varphi_j(x(t_j))-\varphi_j(y(t_j))|]$$

$$+\sum_{i=2}^{k}\sum_{j=1}^{i-1}(t_i-t_{i-1})\left({}_{t_{j-1}}I_{q_{j-1}}^{1}|f(s,x(s)-f(s,y(s))|(t_k)+|\varphi_j(x(t_j))-\varphi_j(y(t_j))|\right)$$

$$+\sum_{i=1}^{k}\left({}_{t_{i-1}}I_{q_{i-1}}^{2}|f(s,x(s)-f(s,y(s))|(t_i)+|\varphi_i^*(x(t_i))-\varphi_i^*(y(t_i))|\right)\}$$

$$+\frac{(t-t_k)^{\gamma+\alpha-1}}{\Gamma_{q_k}(\alpha)}\{\left|\frac{\beta}{1-\beta}\right|[{}_{t_{k_0}}I_{q_{k_0}}^{1}|f(s,x(s))-f(s,y(s))|(\eta)$$

$$+\sum_{j=1}^{k_0} {}_{t_{j-1}}I_{q_{j-1}}^{1}|f(s,x(s)-f(s,y(s))|(t_j)+\sum_{j=1}^{k_0}|\varphi_j(x(t_j))-\varphi_j(y(t_j))|]$$

$$+\sum_{j=1}^{k} {}_{t_{j-1}}I_{q_{j-1}}^{1}|f(s,x(s)-f(s,y(s))|(t_j)+\sum_{j=1}^{k}|\varphi_j(x(t_j))-\varphi_j(y(t_j))|\}$$

$$+(t-t_k)^{\gamma}\, {}_{t_k}I_{q_k}^{\alpha}|f(s,x(s))-f(s,y(s)|(t)$$

$$\le \{\frac{(t-t_k)^{\alpha-2}}{\Gamma_{q_k}(\alpha-1)}\left[\left|\frac{\beta T}{1-\beta}\right|(N_1\eta+k_0N_2)+kN_3+2N_1kT^2+\frac{k^2TN_2}{2}\right]$$

$$+\frac{(t-t_k)^{\alpha-1}}{\Gamma_{q_k}(\alpha)}\left[\left|\frac{\beta}{1-\beta}\right|(N_1\eta+k_0N_2)+N_1T+kN_2\right]+\frac{N_1T^{\alpha}}{\Gamma_{q_k}(\alpha+1)}\}\|x-y\|_{PC_{\gamma}}$$

$$\leq \frac{T^*}{\Gamma^*}[\left|\frac{\beta}{1-\beta}\right|(N_1\eta + k_0N_2)(T+1) + kN_3 + N_1(1+T+2kT^2) + N_2(k+\frac{k^2T}{2})]\|x-y\|_{PC_\gamma}$$
$$\leq \frac{T^*}{\Gamma^*}[\left|\frac{\beta}{1-\beta}\right|(N_1\eta + mN_2)(T+1) + mN_3 + N_1(1+T+2mT^2) + N_2(m+\frac{m^2T}{2})]\|x-y\|_{PC_\gamma}$$
$$< \Lambda\|x-y\|_{PC_\gamma}$$

where Λ is given by (3.3). Thus, $\|Ax-Ay\|_{PC_\gamma} \leq \Lambda\|x-y\|_{PC_\gamma}$. As $\Lambda<1$, therefore, A is a contraction. Hence, by the contraction mapping principle, equation (1.1) has a unique solution.

Examples

Example 4.1. Consider the following impulsive fractional q-difference initial value problem:

$$\begin{cases} {}_{t_k}D^{\frac{3}{2}}_{\left(\frac{k^3-3k+7}{2k^4+k+8}\right)}x(t) = t^2\arctan^2 x(t) + e^t x^3(t),\ t\in[0,\frac{11}{10}], t\neq t_k, \\ \Delta x(t_k) = k - k\cos x(t_k),\ k=1,2,\ldots,10, t_k=\frac{k}{10}, \\ \Delta^* x(t_k) = k\sin^3 x(t_k), k=1,2,\ldots,10, t_k=\frac{k}{10}, \\ x(0)=0,\ {}_0D^{\frac{1}{2}}_{\frac{7}{8}}x(0) = \frac{2}{3}\,{}_{\frac{1}{5}}D^{\frac{1}{2}}_{\frac{3}{14}}x(\frac{1}{4}), \end{cases}$$

Here $\alpha=3/2$, $q_k=(k^3-3k+7)/(2k^4+k+8)$, $k=1,2,\ldots,10$, $m=10$, $T=11/10$, $\beta=2/3$, $k_0=2$, $\eta=1/4$, $f(t,x(t))=t^2\arctan^2 x(t)+e^t x^3(t)$, $\varphi_k(x(t_k))=k-k\cos x(t_k)$, $\varphi_k^*(x(t_k))=k\sin^3 x(t_k)$,

Clearly, all the assumptions of Theorem 3.1 are satisfied. Thus, by the conclusion of Theorem 3.1, the impulsive fractional q-difference initial value problem 4.1 has at least one solution.

Example 4.2. Consider the following impulsive fractional q-difference initial value problem:

$$\begin{cases} {}_{t_k}D^{\frac{3}{2}}_{\left(\frac{k^3-3k+7}{2k^4+k+8}\right)}x(t) = \frac{e^t\sin^5 x(t)}{1+x^4(t)},\ t\in[0,1], t\neq t_k, \\ \Delta x(t_k) = k + 3k\cos^2 x(t_k),\ k=1,2,\ldots,9, t_k=\frac{k}{10}, \\ \Delta^* x(t_k) = k\sin(4+e^{x(t_k)}), k=1,2,\ldots,9, t_k=\frac{k}{10}, \\ x(0)=0,\ {}_0D^{\frac{1}{2}}_{\frac{7}{8}}x(0) = \frac{2}{3}\,{}_{\frac{1}{5}}D^{\frac{1}{2}}_{\frac{3}{14}}x(\frac{1}{4}), \end{cases}$$

Here $\alpha=3/2$, $q_k=(k^3-3k+7)/(2k^4+k+8)$, $k=1,2,\ldots,9$, $m=9$, $T=1$, $\beta=2/3$, $k_0=2$, $\eta=1/4$,
$f(t,x(t))=\frac{e^t\sin^5 x(t)}{1+x^4(t)}$, $\varphi_k(x(t_k))=k+3k\cos^2 x(t_k)$, $\varphi_k^*(x(t_k))=k\sin(4+e^{x(t_k)})$,

Clearly $L_1=e, L_2=36, L_3=9$ and the conditions of Theorem 3.2 can readily be verified. Therefore, the conclusion of Theorem 3.2 applies to the impulsive fractional q-difference initial value problem 4.2.

REFERENCES

[1]Jackson, FH: q-Difference equations. Am. J. Math. 32, 305-314 (1910).

[2] Al-Salam, WA: Some fractional *q*-integrals and *q*-derivatives. Proc. Edinb. Math. Soc. 15(2), 135-140 (1966/1967).
[3] Agarwal, RP: Certain fractional *q*-integrals and *q*-derivatives. Proc. Camb. Philos. Soc. 66, 365-370 (1969).
[4] Bangerezako, G: Variational *q*-calculus. J. Math. Anal. Appl. 289, 650-665 (2004).
[5] Dobrogowska, A, Odzijewicz, A: Second order *q*-difference equations solvable by factorization method. J. Comput.Appl. Math. 193, 319-346 (2006).
[6]Gasper, G, Rahman, M: Some systems of multivariable orthogonal *q*-Racah polynomials. Ramanujan J. 13, 389-405(2007).
[7]Ismail, MEH, Simeonov, P: *q*-Difference operators for orthogonal polynomials. J. Comput. Appl. Math. 233, 749-761(2009).
[8] Bohner, M, Guseinov, GS: The *h*-Laplace and *q*-Laplace transforms. J. Math. Anal. Appl. 365, 75-92 (2010).
[9]El-Shahed, M, Hassan, HA: Positive solutions of *q*-difference equation. Proc. Am. Math. Soc. 138,1733-
1738 (2010).
[10]Ahmad, B: Boundary-value problems for nonlinear third-order *q*-difference equations. Electron. J. Differ. Equ. 2011, 94(2011).
[11] Ahmad, B, Alsaedi, A, Ntouyas, SK: A study of second-order *q*-difference equations with boundary conditions. Adv.Differ. Equ. 2012, 35 (2012).
[12]Ahmad, B, Nieto, JJ: On nonlocal boundary value problems of nonlinear *q*-difference equations. Adv. Differ. Equ.2012, 81 (2012).
[13]Yu, C, Wang, J: Existence of solutions for nonlinear second-order *q*-difference equations with first-order *q*-derivatives. Adv. Differ. Equ. 2013, 124 (2013).
[14]Tariboon, J, Ntouyas, SK: Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. 2013, 282 (2013).
[15] C. Thaiprayoon , J. Tariboon, S.K. Ntouyas: Separated boundary value problems for second-order impulsive q-integro-difference equations, Adv. Differ. Equ. 2014 ,88 (2014) .
[16]J. Tariboon , S.K. Ntouyas , P. Agarwal: New concepts of fractional quantum calculus and applications to impulsive fractional *q* -difference equations. Adv. Differ. Equ. 2015,18 (2015).
[17]Bashir Ahmad , Sotiris K. Ntouyas , Jessada Tariboon, Ahmed Alsaedi , Hamed H. Alsulami: Impulsive
fractional *q* -integro-difference equations with separated boundary conditions. Applied Mathematics and Computation, 281,199–213(2016) .
[18]J.X. Sun:Nonlinear Functional Analysis and its Application, Science Press, Beijing, 2008.