ON AUTOMATIC CONTINUITY OF DERIVATIONS FOR BANACH ALGEBRAS WITH INVOLUTION

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ABSTRACT

We treat the problem of the automatic continuity of the Derivations in Banach algebras provided with an involution σ. To do this, we introduce and study on a unitary algebra provided with an involution a notion which we call σ-semi-simplicity. It is based on the study of certain bilateral ideals called σ-ideals.

Keywords: σ-Algebra, σ-Simple algebra, σ-Semi-Simple Algebra, Automatic Continuity, Separating ideal.

INTRODUCTION

In Automatic Continuity theory we are concerned with algebraic conditions on a linear map between Banach spaces which make this map automatically continuous. This theory has been mainly developed in the context of Banach algebras, and there are excellent accounts on automatic continuity theory [2, 3, 5] (see also [6]) in this associative context. In [7] Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the Jacobson radical. They conjectured that the assumption of continuity is unnecessary. In [4] Johnson proved that if A is a semi-simple Banach algebra, then every derivation on A is continuous and hence by the Singer-Wermer theorem it is zero.

In this work, we define and study on a unitary algebra provided with an involution σ a notion which called σ-semi-simplicity which generalizes the notion of semi-simplicity, it rests on the study of certain bilateral ideals called σ-ideals. The interest therefore is to restrict oneself to the level of a family of bilateral ideals instead of considering all the ideals on the left. This notion of σ-semi-simplicity will also contribute to the study of the automatic continuity of linear operators on Banach algebras, in particular the continuity of derivations. We will show that on a σ-semi-simple Banach algebra, every derivation is continuous (Theorem 2.2).

Preliminaries

In these papers, the algebras considered are assumed complex, Unitary, not necessarily commutative. An involution σ on an algebra A is a mapping: satisfying the following properties: $\sigma(x+y)=\sigma(x)+\sigma(y)$, $\sigma(xy)=\sigma(y)\sigma(x)$, $\sigma(\sigma(x))=x$, $\sigma(\lambda x)=\bar{\lambda}\sigma(x)$ $\forall\lambda \in$ IK, for all x, y in A. With involution σ, A is called σ-algebra. An ideal I of σ-algebra is called a σ-ideal if $\sigma(I) \subseteq I$ (then $\sigma(I) = I$). Moreover, I is said to be a *σ-minimal* (resp. *σ-maximal*) *ideal of* A if I is minimal (resp. maximal) in the set of nonzero (resp. proper) σ -ideals of A. Observe that if I is an ideal of A, then $I+\sigma(I)$, $I\ \sigma(I)$, $\sigma(I)I$ and $I \cap \sigma(I)$ are σ-ideals of A. Moreover, if we denoted by $\overline{\sigma}$ the map from A/I to A/I defined by $\overline{\sigma}(a+I) = \sigma(a)+I$, then $\overline{\sigma}$ is a well-defined involution on A/I.

Characterizations of σ-semi-simple algebras

An algebra A is called *simple* if it has no proper ideals. An *σ-algebra* A is called *σ-simple* if it has no proper *σ-ideals*. We observe that every *simple* algebra with involution (A,σ) is a *σ-simple*. The following counterexample shows the converse is not true.

Counterexample 1.1 Let A be a simple algebra, we denoted by $A°$ the opposite algebra A. Consider the algebra $B = A \oplus A°$. Provided with the exchange involution defined by: $\sigma(x,y) = (y,x)$, It clear that B is not simple, since the ideals of B are *(0)*, A, *{0}*x $A°$ and Ax*{0}*. But B is σ -simple. Indeed, the only σ-ideals of B are *0* and B.
It is therefore natural to ask under what conditions the converse is true. It is subject to the following proposition:

Proposition 1.1 *Let (A,σ) be σ -simple algebra. If the involution σ is anisotropic, then A is simple.*
Recall that involution is called anisotropic if : $\forall a \in A$, it $\sigma(a)a = 0\ = 0 \Rightarrow a\ = 0$.

Proof
Let I be an ideal of A, then $I \cap \sigma(I)$ is an σ-ideal. It follows that, $I \cap \sigma(I) = \{0\}$ or $= A$.
If $I \cap \sigma(I) = \{0\}$, then $\sigma(x)x = 0\ \forall x \in I$. Since σ is anisotropic, then $x = 0$, a result that $I = \{0\}$. If $I \cap \sigma(I) = A$, then $I = A$ ■

Proposition 1.2 *Let A is σ-algebra. Then A is a σ -simple if, and only if, there exist a maximal ideal M such that, $M \cap \sigma(M) = \{0\}$.*

Proof
$\Rightarrow$ We assume A is σ-simple. Let M be a maximal ideal of A. We have $M \cap \sigma(M)$ is a σ -ideal of A, then $M \cap \sigma(M) = \{0\}$ or A. If $M \cap \sigma(M) = A$, then $M = A$, which contradicts the fact that M is a proper ideal. Hence, $M \cap \sigma(M) = \{0\}$.
$\Leftarrow$ Assume that, there exists a maximal ideal M such that $M \cap \sigma(M) = \{0\}$. Let I is a σ-ideal of A. If $I \subseteq M$, then $\sigma(I) = I \subseteq \sigma(M)$, where $I \subseteq M \cap \sigma(M) = \{0\}$. If $I \not\subset M$, then $M = I + A$, and we have: $\sigma(M) + I = (\sigma(M) + I)A = (\sigma(M) + I)(M + I) \subseteq \sigma(M)M + I = I$. Which implies that $\sigma(M) \subseteq I$, as a result, $M \subset I$. Since M is maximum ideal of A, so it follows that $A = I$ ■

Proposition 1.3 [8] *Let A an σ-simple algebra which is not simple. Then, there exists a sub- algebra simple unit I of A such that $A = I \oplus \sigma(I)$.*

Proof
Let I a proper ideal of A. So it follows that $I \cap \sigma(I)$ is a σ-ideal, since A is a σ-

simple algebra, then $I \cap \sigma(I) = \{0\}$ or $I \cap \sigma(I) =$ A. If $I \cap \sigma(I) = A$ then $I = A$, which is absurd. From where $I \cap \sigma(I) = \{0\}$. There is also $I + \sigma(I)$ is a σ-ideal, then $I + \sigma(I) = \{0\}$, or $I + \sigma(I) = A$. If $I + \sigma(I) = \{0\}$, then $I = \{0\}$, which contradicts the fact that I is *proper*. Therefore, $A = I \oplus \sigma(I)$. Let J an ideal of A such that $J \subseteq I$. According to what precedes, $A = J \oplus \sigma(J)$. Let $i \in I$, then there exists $j, j' \in J$ such that $i = j + \sigma(j')$. However $i - j = \sigma(j') \in I \cap \sigma(I) = \{0\}$, from where $i = j$, therefore $I = J$. Consequently, I is a minimal ideal of A. Let J an ideal of I, then J is an ideal of A. Indeed, let $a \in A$ and $j \in J$, then it exists $i, \sigma(i')$ such that $a = i + \sigma(i')$. From where $aj = (i + \sigma(i'))$. $j = ij + \sigma(i')j$. However, $\sigma(i')j \in \sigma(I)I$ and $\sigma(I)I \subseteq I \cap \sigma(I) = \{0\}$, consequently $aj = ij \in I$. Since I is a minimal ideal, then $J = \{0\}$ or $I = J$. Thus, I a simple sub-algebra. On other hand, I a unital and if 1 indicates the unit of A, then there exists $e, e' \in I$ such that $1 = e + \sigma(e')$. Let $x \in I$, we are: $x = x1 = xe + x\sigma(e')$, but $x - xe = x\sigma(e') \in I \cap \sigma(I) = \{0\}$, from where $x = xe$. In the same way, we checked that $x = ex$. Consequently, I a unital of unit e ■

Proposition 1.4 Let A be a σ-algebra and M σ-maximal ideal which is not maximal. Then there exists a maximal ideal N of A such that $M = N \cap \sigma(N)$.

Proof

As M is not maximal, there is a maximal ideal N of A such that $M \subset N$. Since $\sigma(M) = M \subset \sigma(N)$, where $M \subset N \cap \sigma(N)$. Since $N \cap \sigma(N)$ is a σ-ideal of A, it follows therefore that $M = N \cap \sigma(N)$ ■

Definition 1.1 Let A be a σ-algebra. We call σ-radical of A, denoted $Rad_\sigma(A)$, the intersection of all ideals σ-maximal of A. A is called σ-semi-simple if $Rad_\sigma(A) = \{0\}$.

Proposition 1.5 Let I be a σ-ideal of a σ-algebra A such that $I \subseteq Rad_\sigma(A)$. So $Rad_\sigma(A/I) = Rad_\sigma(A)/I$ In particular, $A/Rad_\sigma(A)$ is a σ-semi simple.

Proof

M is a σ-maximal ideal of A. We put $\overline{A} = A/I$ and $\overline{M} = M/I$. We have: $I \subseteq Rad_\sigma(A) \subseteq M$. So from the following canonical isomorphism: $\overline{A}/\overline{M} \approx A/M$ which is σ-simple, it follows that $\overline{A}/\overline{M}$ is a σ-simple algebra. Consequently, M/I is a σ-ideal σ-maximal of A/I. From where:

$$Rad_\sigma(A/I) = \cap\{\overline{M} : M \text{ is } \sigma\text{-maximal ideal of } A\}$$
$$= \overline{\cap\{M : M \text{ is } \sigma\text{-maximal ideal of } A\}}$$
$$= \overline{Rad_\sigma(A)} = Rad_\sigma(A)/I$$ ■

Now, we say that an algebra with involution (A, σ) is σ-*semi-simple* if A is a sum of σ-minimal ideals of A.

Lemma 1.1 *Let* A *be a* σ*-semi-simple algebra such that* $A=\sum_{i\in S} I_i$*, where each* I_i *is a* σ*-minimal ideal of* A*. If* P *is a* σ*-minimal ideal of* A*, then there is a subset* T *of* S *such that:* $A = P \oplus (\oplus_{j\in T} I_j)$

Proof. Since I_i are σ-minimal and $P \neq I$, then there exists some $i \in S$ such that $I_i + P$ is a direct sum. Indeed, otherwise $I_i \cap P = I_i$ for all $i \in S$, which implies that $P = A$.
Applying Zorn's lemma, there is a subset T of S such that the collection $\{ I_i : i \in T \} \cup \{P\}$ is maximal with respect to independence: $(\oplus_{j\in T} I_j) + P = (\oplus_{j\in T} I_j) \oplus P$.
Setting B $= (\oplus_{j\in T} I_j) + P$, the maximality of T implies that $I_i \cap B \neq (0)$ for all $i \in S$. Then, the σ-minimality of I_i yields that $I_i \cap B = I_i$ hence $I_i \subseteq B$ for all $i \in S$. Consequently $B = A$.■

Corollary1.1 *For an algebra with involution* (A,σ)*, the following conditions are equivalent:*
1) *A is a* σ*-semi-simple.*
2) *A is a direct sum of* σ*-minimal ideals*

Example.1.2 Let A_4 be the alternating group on 4 letters. Consider the group algebra *IR* $[A_4]$ provided with its canonical involution σ defined by:

$$\sigma\left(\sum_{g\in A_4} r_g g\right) = \sum_{g\in A_4} r_g g^{-1}$$

From[1], the decomposition of the semi-simple algebra $IR[A_4]$ into a direct sum of simple components is as follows: $IR[A_4] = B_1 \oplus B_2 \oplus B_3$, where each B_i is invariant under σ. More explicitly, $B_1 \approx IR$, $B_2 \approx C$, and $B_3 \approx M_3(IR)$. In particular, each B_i is a σ-minimal ideal of $IR[A_4]$. Consequently, $IR[A_4]$. is a σ-semi-simple algebra.

Now, let A be a σ - A -simple algebra. Since A is finitely generated (indeed, 1 generates A), then A has a finite length. Thus $A = \oplus_{i=1}^{l} I_i$, where each I_i is a σ -minimal ideal of A. It is easy to verify that each I_i is generated by a central symmetric idempotent element $e_i \in A$ (i.e: $e_i^2 = e_i$ and $\sigma(e_i) = e_i$), where $1 = \sum_{i=1}^{l} e_i$. Moreover, $e_i e_j = 0$ for all $i \neq j$. In what follows, we denote by S the set of central symmetric orthogonal idempotents of A, i.e. $S = \{e_1 ....e_l\}$ such that $I_i = Ae_i$.

Let $A = \oplus_{i=1}^{l} I_i$ be a σ -semi-simple algebra, we have already seen that each I_i is generated by a central symmetric idempotent e_i such that $1 = \sum_{i=1}^{l} e_i$. Hence, I_i is a subalgebra of A with unity e_i. Moreover, I_i is a σ -simple algebra for all $1 \le i \le l$. Consequently, every σ -semi-simple algebra is a direct sum of σ -simple algebras.

Automatic Continuity

A derivation D *on algebra* A is linear mapping from A to itself satisfying $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$

Let D a derivation of a Banach space X. Then, the separating ideal $\delta(D)$ of X is the subset of X defined by: $\delta(D) = \{ y \in X / \exists (x_n)_n \subset X : x_n \to 0$ and $D(x_n) \to y\}$

Lemma 2.1 [6]

Let S be a linear operator from a Banach space X into a Banach space Y. Then;

i) $\delta(S)$ is a closed linear space of Y

ii) S is continuous if only if $\delta(S)$ ={0} and

iii) If T and R are continuous linear operators on X and Y respectively, and if $ST = RS$, then $R\delta(S) \subset \delta(S)$

Lemma 2.2 [6]

Let S be a linear operator from a Banach space X into a Banach space Y, and let R be a continuous operator from Y into a Banach space Z. Then:

i) RS is continuous if and if $R\delta(S) = \{0\}$.

ii) $\overline{R\delta(S)} = \delta(RS)$, and

iii) There is a constant M (independent of R and Z) such that if RS is continuous then $\|RS\| \le M\|R\|$

Proposition 2.1 *Let A be a Banach σ -algebra A, then a σ -maximal σ -ideal M of A is closed.*

Proof
If M is a maximal ideal of A, then M is closed,. Otherwise, if M not Maximal, there is a maximal ideal N of A such that $M = N \cap \sigma(N)$ (proposition 1.4). Since N (resp. $\sigma(N)$) is closed, it is deduced that M is closed in A ■

Proposition 2.2 Let A a Simple Banach algebra. Then all *dérivation* D on A *is continuous.*
Proof
Let $\delta(D)$ the separator ideal of D in A is simple, so $\delta(D) = \{0\}$ or $\delta(D) = A$. If $\delta(D) = A$, that $e_A \in \delta$(D), cconsequently $0 \in Sp(e_A)$ ([6] theorem 6-16). From where $\delta(D) = \{0\}$. And by Lemma 2.1, as a result, D is continuous ■

Theorem 2.1
Let A a σ-Simple Banach σ-algebra . Then all *dérivation* D on A *is continuous.*

Proof We have A is an algebra σ simple, there exists simple unital subalgebra I of A such that : $A = I \oplus \sigma(I)$ (Proposition 1.3); following algebraic isomorphism: $I \approx A/\sigma(I)$, one deduces that I am a maximal ideal of A . From where I (resp; $\sigma(I)$) is closed in A. Consequently, the algebra A/I (resp; $A/\sigma(I)$) is a simple Banach σ-algebra. Since I is an ideal of A, then so is $D(I)+I$; therefor $D(I)+I/I$ is an ideal of A/I . As A/I is a simple algebra, so $D(I)+I/I = \{\bar{0}\}$ or $D(I)+I/I = A/I$. Since I I is a maximal ideal of A, then $D(I)+I = I$, so $D(I) \subseteq I$. Consider the function $\tilde{D}$ on A/I defined by: $\tilde{D}(a+I) = D(a)+I$. We show that is a derivation on A/I . Note that it is easy to show $\tilde{D}$ is linear operator. Moreover, for $a,b \in A$, $\tilde{D}((a+I)(b+I)) = \tilde{D}(ab+I) = D(ab)+I = aD(b)+D(a)b+I$.
But then, $(a+I)\tilde{D}(b+I) + \tilde{D}(a+I)(b+I) = (a+I)(D(b)+I)+(D(a)+I)(b+I) = aD(b)+I+D(a)b+I = aD(b)+D(a)b+I$. So $\tilde{D}$ is a derivation on the simple Banach algebra A/I, then by proposition 2.2, $\tilde{D}$ is continuous. To show that D is continuous, consider the canonical surjection $\pi : A \to A/I$; $a \to a+I$ which is continuous. To show that D is continuous, we observe first that $\pi \circ D = \tilde{D} \circ \pi$ because for every a $\in A$, we have $\pi \circ D(a) = \pi(D(a)) = D(a)+I$ and $\tilde{D}\,\pi(a) = \tilde{D}(a+I) = D(a)+I$. Since $\tilde{D} \circ \pi$ is continuous, then; we have $\delta(\tilde{D} \circ \pi) = \{\bar{0}\}$, And $\overline{\pi\delta(D)} = \delta(\tilde{D} \circ \pi) = \{\bar{0}\}$ (Lemma 2.2) and this implied that $\delta(D) \subset I$. Following the same steps, we show that $\delta(D) \subset \sigma(I)$, then $\delta(D) \subset I \cap \sigma(I) = \{0\}$. Therefore D is continuous (lemma 2.1). ■

Theorem 2.2
Let A a σ*-semi*--Simple Banach σ-algebra . Then all *dérivation* D on A *is continuous.*

Proof
Since A is a σ-semi-simple algebra, writing (by lemma 1.1) $A = \oplus_{i=1}^{l} I_i$ where I_i is a σ-minimal ideal of A and setting $L_i = \oplus_{j \neq i} I_j$, then $\forall 1 \leq i \leq l$ L_i is a σ-maximal ideal of A.
If L_i is a maximal ideal, then $D(L_i) \subseteq L_i$ If L_i is not maximal, then by proposition 1.4, that exist a maximal ideal N_i such that $\forall 1 \leq i \leq l$, $L_i = N_i \cap N^*{}_i$. Consequently,

$D(L_i) = D(N_i \cap \sigma(N)_i) \subset D(N_i) \cap D(\sigma(N)_i) \subset N_i \cap \sigma(N)_i = L_i \ \forall 1 \le i \le l$. Now, consider the function $\tilde{D}$ on A/I defined by: $\forall 1 \le i \le l \ \tilde{D}\ (a + L_i) = D(a) + L_i$. Since a σ -maximal ideal is closed (proposition (2.1) and as mentioned in theorem (2.1), we have $\tilde{D}$ is a derivation on the σ -simple Banach algebra A/L_i . then by theorem 2.1, we have $\tilde{D}$ is continuous. Consider the canonical surjection $\pi : A \to A/L_I$; $a \to a + L_i$ which is continuous. To show that D is continuous, we observe first that π o D = $\tilde{D}$ o π because for every a $\in A$, we have π o $D(a)$ = $\pi(D(a))$ = $D(a) + L_i$ and $\tilde{D}\ \pi(a)$= $\tilde{D}(a + L_i)$ = $D(a) + L_i$. Since $\tilde{D}$ o π is continuous, then; we have $\delta(\tilde{D}$ o π) = { $\bar{0}$ }, And $\overline{\pi\delta(D)}$ = $\delta(\tilde{D} o \pi)$ = { $\bar{0}$ } and this implied that $\delta(D) \subset L_i \ \forall 1 \le i \le l$. Thus implied that $\delta(D) \subset \cap_{i=1}^{l} L_i = \{0\}$. Consequently, D is continuous ■

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