NONLOCAL BOUNDARY VALUE PROBLEM FOR SECOND ORDER ANTI-PERIODIC NONLINEAR IMPULSIVE q_k-INTEGRODIFFERENCE EQUATION

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ABSTRACT

A second order anti-periodic nonlinear impulsive integrodifference equation within the frame of q_k-quantum calculus is investigated by applying using fixed point theorems. The conditions for existence and uniqueness of solution are obtained.

KEYWORDS: q_k-Integrodifference Equation, q_k-derivatives, q_k-integrals, Boundary Value Problem.

INTRODUCTION

The $q-$calculus was initiated in twenties of the last century. However, it has gained considerable popularity and importance during the last three decades or so. Their study has not only important theoretical meaning but also wide applications in conformal quantum mechanics, high energy physics, etc. We refer the reader to recent articles [1-7]. Recently, in [8], authors research first order nonlocal boundary value problem for nonlinear impulsive q_k-integrodifference equation and in [9], authors research existence of solutions for a class of anti-periodic boundary value problems with fractional *q*-difference equation.

On this line of thought in this paper, we study the existence and uniqueness of solutions for second order nonlinear q_k-integrodifference equation with nonlocal boundary condition and impulses:

$$\begin{aligned}
&D_{q_k}{}^2u(t) = f(t,u(t)) + {}_{t_k}I_{q_k}g(t,u(t)), \qquad 0<q_k<1, t\in J', \\
&\Delta u(t_k) = I_k(u(t_k)), \qquad t_k\in(0,1), \\
&D_{q_k}u(t_k^+) - D_{q_{k-1}}u(t_k^-) = L_k(u(t_k^-)), \qquad k=1,2,\cdots,p,p+1, \qquad (1) \\
&u(0^+) = -u(1^-), \\
&D_{q_0}u(0^+) = -D_{q_{p+1}}u(1^-),
\end{aligned}$$

where D_{q_k}, ${}_{t_k}I_{q_k}$ are q_k-derivatives and q_k-integrals $(k=0,1,\cdots,p+1)$, respectively. $f,g\in C(J\times \mathrm{R},\mathrm{R})$, $I_k,h\in C(\mathrm{R},\mathrm{R})$, $J=[0,1]$, $0=t_0<t_1<t_2<\cdots<t_p<t_{p+1}=1$, $J'=[0,1]\setminus\{t_1,t_2,\cdots,t_p\}$, $\Delta u(t_k)=u(t_k^+)-u(t_k^-)$.

Where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t=t_k (t=1,2,\cdots,p)$, respectively.

PRELIMINARIES

Let us set $J_0=[0,t_1], J_1=(t_1,t_2],\cdots,J_{p-1}=(t_{p-1},t_p], J_p=(t_p,1]$ and introduce the space: $PC(J,\mathrm{R})=\{u:J\to\mathrm{R}\,|\,u\in C(t_k,t_{k+1}],k=0,1,\cdots,p,\text{and } x(t_k^+) \text{ and } x(t_k^-) \text{ exist}$ with $x(t_k^-)=x(t_k),k=1,2,\cdots,p\}.$ (2)

And

$PC^1(J,\mathrm{R})=\{u\in PC(J,\mathrm{R}),D_{q_k}(t_k^+),D_{q_k}(t_k^-)$ exist and $D_{q_k}x(t)$ is left continuous at t_k, for $k=1,2,\cdots,p\}$, where $J=[0,1]$, note that $PC^1(J,\mathrm{R})$ is a Banach space with the norm $\|u\|_{PC^1}=\sup_{t\in J}\left\{\|u\|_{PC},\|D_{q_k}u\|_{PC}\right\}.$ (3)

Definition a function $u\in PC^1(J,\mathrm{R})$ with its derivative of second order existing on J is a solution of (1) if it satisfies (1).

For convenience, let us recall some basic concepts of q_k-calculus (J. Tariboon et al, 2013).
For $0<q_k<1$ and $t\in J_k$, we define the q_k-derivatives of a real valued continuous function f as

$$D_{q_k}f(t)=\frac{f(t)-f(q_kt+(1-q_k)t_k)}{(1-q_k)(t-t_k)},$$

$$D_{q_k}f(t_k)=\lim_{t\to t_k}D_{q_k}f(t). \tag{4}$$

Higher order q_k-derivatives are given by

$$D_{q_K}^0f(t)=f(t),\quad D_{q_K}^nf(t)=D_{q_k}D_{q_k}^{n-1}f(t),\quad n\in N,t\in J_k. \tag{5}$$

The q_k-integral of a function f is defined by

$$_{t_k}I_{q_k}f(t):=\int_{t_k}^t f(s)\,d_{q_k}s=(1-q_k)(t-t_k)\times\sum_{n=0}^{\infty}q_k^nf\left(q_k^nt+(1-q_k^n)t_k\right),\quad t\in J_k. \tag{6}$$

Provided the series converges. If $a\in(t_k,t)$ and f is defined on the interval (t_k,t), then

$$\int_a^t f(s)\,d_{q_k}s=\int_{t_k}^t f(s)\,d_{q_k}s-\int_{t_k}^a f(s)\,d_{q_k}s. \tag{7}$$

Observe that

$$D_{q_k}\left({}_{t_k}I_{q_k}f(t)\right)=D_{q_k}\int_{t_k}^t f(s)d_{q_k}s=f(t),$$

$$_{t_k}I_{q_k}\left(D_{q_k}f(t)\right)=\int_{t_k}^t D_{q_k}f(s)d_{q_k}s=f(t)-f(t_k), \tag{8}$$

$$_aI_{q_k}\left(D_{q_k}f(t)\right)=\int_a^t D_{q_k}f(s)d_{q_k}s=f(t)-f(a),\quad a\in(t_k,t).$$

For $t\in J_k$, the following reversing order of q_k-integration holds

$$\int_{t_k}^t\int_{t_k}^s f(r)d_{q_k}rd_{q_k}s=\int_{t_k}^t\int_{q_kr+(1-q_k)t_k}^t f(r)d_{q_k}sd_{q_k}r. \tag{9}$$

Note that if $t_k=0$ and $q_k=q$ in (4) and (6). Then $D_{q_k}f=D_qf, {}_{t_k}I_{q_k}f={}_0I_qf$, where D_q and ${}_0I_q$ are the well-known $q-$derivative and $q-$integral of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad (10)$$

$${}_0I_q f(t) = \int_0^t f(s) d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n).$$

Lemma 1. For given $y_{q_k} \in C(J, \mathrm{R})$, the function $u \in PC^1(J, \mathrm{R})$ is a solution of the impulsive
q_k − integrodifference equation

$$\begin{aligned} &D_{q_k}^2 u(t) = y_{q_k}(t), \quad 0 < q_k < 1, t \in J', \\ &D_{q_k} u(t_k^+) - D_{q_{k-1}} u(t_k^-) = L_k(u(t_k)), \\ &\Delta u(t_k) = I_k(u(t_k^-)), \quad k = 1,2,\cdots, p, \\ &u(0^+) = -u(1^-), \\ &D_{q_0} u(0^+) = -D_{q_{p+1}} u(1^-), \end{aligned} \qquad (11)$$

if and only if u satisfies the q_k − integral equation

$$u(t) = \begin{cases} \left(\frac{1}{4} + \frac{1}{4}t_0 - \frac{1}{2}t\right)\left[\sum_{i=1}^{p} L_i(u(t_i)) + \sum_{i=0}^{p} \int_{t_i}^{t_{i+1}} y_{q_i}(s) d_{q_i} s\right] + \int_{t_0}^{t} [(t-t_0) - q_0(s-t_0)] y_{q_0}(s) d_{q_0} s \\ \quad -\frac{1}{2}\int_{t_0}^{1} [(1-t_0) - q_0(s-t_0)] y_{q_0}(s) d_{q_0} s & t \in J_0, \\ \left(\frac{1}{4} + \frac{1}{4}t_0 - \frac{1}{2}t\right)\left[\sum_{i=1}^{p} L_i(u(t_i)) + \sum_{i=0}^{p} \int_{t_i}^{t_{i+1}} y_{q_i}(s) d_{q_i} s\right] + \frac{1}{2}\sum_{i=1}^{k} I_i(u(t_i)) + \sum_{i=1}^{k}\left(t - \frac{1}{2}t_i - \frac{1}{2}\right) L_i(u(t_i)) \\ \quad + \frac{1}{2}\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} [(t_{i+1} - t_i) - q_i(s-t_i)] \int_{t_i}^{t_{i+1}} y_{q_i}(s) d_{q_i} s + \sum_{i=0}^{k-1}\left(t - \frac{1}{2}t_{i+1} - \frac{1}{2}\right)\int_{t_i}^{t_{i+1}} y_{q_i}(s) d_{q_i} s \\ \quad + \int_{t_k}^{t} [(t-t_k) - q_k(s-t_k)] y_{q_k}(s) d_{q_k} s - \frac{1}{2}\int_{t_k}^{1} [(1-t_k) - q_k(s-t_k)] y_{q_k}(s) d_{q_k} s & t \in J_k. \end{cases}$$

（12）

Proof. Let u be a solution of q_k − difference equation (11). For $t \in J_0$, applying the operator ${}_0I_{q_0}$ on both sides of $D_{q_0}^2 u(t) = y_{q_0}(t)$,
we have

$$\begin{aligned} &D_{q_0} u(t) = D_{q_0} u(0^+) + {}_0I_{q_0} y_{q_0}(t) = D_{q_0} u(0^+) + \int_{t_0}^{t} y_{q_0}(s) d_{q_0} s, \\ &u(t) = u(0^+) + (t - t_0) D_{q_0} u(0^+) + \int_{t_0}^{t} [(t-t_0) - q_0(s-t_0)] y_{q_0}(s) d_{q_0} s. \end{aligned} \qquad (13)$$

Thus,

$$\begin{aligned} &D_{q_0} u(t_1^-) = D_{q_0} u(0^+) + \int_{t_0}^{t_1} y_{q_0}(s) d_{q_0} s, \\ &u(t_1^-) = u(0^+) + (t_1 - t_0) D_{q_0} u(0) + \int_{t_0}^{t_1} [(t_1 - t_0) - q_0(s-t_0)] y_{q_0}(s) d_{q_0} s. \end{aligned} \qquad (14)$$

Similarly, for $t \in J_1$, applying the operator ${}_{t_1^+}I_{q_1}$ on both sides of $D_{q_1}^2 u(t) = y_{q_1}(t)$, then

$$D_{q_1} u(t) = D_{q_1} u(t_1^+) + \int_{t_1}^{t} y_{q_1}(s) d_{q_1} s. \qquad (15)$$

In view of $D_{q_1} u(t_1^+) - D_{q_0} u(t_1^-) = L_1(u(t_1))$ and $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$, it holds

$$D_{q_1}u(t)=D_{q_0}u(0^+)+L_1(u(t_1))+\int_{t_0}^{t_1}y_{q_0}(s)d_{q_0}s+\int_{t_1}^{t}y_{q_1}(s)d_{q_1}s, \tag{16}$$

$$u(t)=u(0^+)+(t-t_0)D_{q_0}u(0^+)+I_1(u(t_1))+(t-t_1)L_1(u(t_1))+(t-t_1)\int_{t_0}^{t_1}y_{q_0}(s)d_{q_0}s$$
$$+\int_{t_0}^{t_1}[(t_1-t_0)-q_0(s-t_0)]y_{q_0}(s)d_{q_0}s+\int_{t_1}^{t}[(t-t_1)-q_1(s-t_1)]y_{q_1}(s)d_{q_1}s. \tag{17}$$

Repeating the above process, we can get

$$D_{q_k}u(t)=D_{q_0}u(0^+)+\sum_{i=1}^{k}L_1(u(t_1))+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}y_{q_{i-1}}(s)d_{q_{i-1}}s+\int_{t_k}^{t}y_{q_k}(s)d_{q_k}s. \tag{18}$$

And

$$\begin{aligned}u(t)&=u(0^+)+(t-t_0)D_{q_0}u(0^+)+\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k}(t-t_i)L_i(u(t_i))\\&+\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}y_{q_{i-1}}(s)d_{q_{i-1}}s+\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}[(t_i-t_{i-1})-q_{i-1}(s-t_{i-1})]y_{q_{i-1}}(s)d_{q_{i-1}}s\\&+\int_{t_k}^{t}[(t-t_k)-q_k(s-t_k)]y_{q_k}(s)d_{q_k}s.\end{aligned} \tag{19}$$

Using the boundary value conditions given in (11), we can get (12).

Conversely, assume that u satisfies the impulsive q_k-integral equation (11); applying D_{q_k} on both sides of (12) and substituting $t=0$ in (12), then (11) holds. This completes the proof.

MAIN RESULTS

Letting $y_{q_k}(t)=f(t,u(t))+{}_{t_k}I_{q_k}g(t,u(t))$, in view of Lemma 1, we introduce an operator $Q:PC^1(J,\mathrm{R})\to PC^1(J,\mathrm{R})$ as

$$\begin{aligned}(Qu)(t)&=\int_{t_k}^{t}[(t-t_k)-q_k(s-t_k)]\left[f(s,u(s))+\int_{t_k}^{s}g(r,u(r))d_{q_k}r\right]d_{q_k}s\\&-\frac{1}{2}\int_{t_k}^{1}[(1-t_k)-q_k(s-t_k)]\left[f(s,u(s))+\int_{t_k}^{s}g(r,u(r))d_{q_k}r\right]d_{q_k}s\\&+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}[(t_{i+1}-t_i)-q_i(s-t_i)]\left[f(s,u(s))+\int_{t_i}^{s}g(r,u(r))d_{q_i}r\right]d_{q_i}s\\&+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\left[f(s,u(s))+\int_{t_i}^{s}g(r,u(r))d_{q_i}r\right]d_{q_i}s\\&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=1}^{p}L_i(u(t_i))+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[f(s,u(s))+\int_{t_i}^{s}g(r,u(r))d_{q_i}r\right]d_{q_i}s\right)\\&+\frac{1}{2}\sum_{i=1}^{k}I_i(u(t_i))+\sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)L_i(u(t_i)).\end{aligned} \tag{20}$$

And

$$\begin{aligned}(D_{q_k}Qu)(t)&=\int_{t_k}^{t}\left[f(s,u(s))+\int_{t_k}^{s}g(r,u(r))d_{q_k}r\right]d_{q_k}s+\sum_{i=1}^{k}L_i(u(t_i))\\&+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left[f(s,u(s))+\int_{t_i}^{s}g(r,u(r))d_{q_i}r\right]d_{q_i}s\end{aligned} \tag{21}$$

$$-\frac{1}{2}\left(\sum_{i=1}^{p} L_i\left(u\left(t_i\right)\right)+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[f\left(s,u(s)\right)+\int_{t_i}^{s} g\left(r,u\left(r\right)\right)d_{q_i}r\right]d_{q_i}s\right).$$

By reversing the order of integration, we obtain

$$\begin{aligned}(Qu)(t) &= \int_{t_k}^{t}[(t-t_k)-q_k(s-t_k)]\left[f\left(s,u(s)\right)+[(t-t_k)-q_k(s-t_k)]\times g\left(s,u(s)\right)\right]d_{q_k}s\\ &-\frac{1}{2}\int_{t_k}^{1}[(1-t_k)-q_k(s-t_k)]\left[f\left(s,u(s)\right)+[(1-t_k)-q_k(s-t_k)]\times g\left(s,u(s)\right)\right]d_{q_k}s\\ &+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}[(t_{i+1}-t_i)-q_i(s-t_i)]\left[f\left(s,u(s)\right)+\left[(t_{i+1}-t_i)-q_i\left(s-t_i\right)\right]\times g\left(s,u(s)\right)\right]d_{q_i}s\\ &+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\left[f\left(s,u(s)\right)+\left[(t_{i+1}-t_i)-q_i\left(s-t_i\right)\right]\times g\left(s,u(s)\right)\right]d_{q_i}s\\ &+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=1}^{p}L_i\left(u\left(t_i\right)\right)+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[f\left(s,u\left(s\right)\right)+\left[(t_{i+1}-t_i)-q_i\left(s-t_i\right)\right]\times g\left(s,u\left(s\right)\right)\right]d_{q_i}s\right)\\ &+\frac{1}{2}\sum_{i=1}^{k}I_i\left(u\left(t_i\right)\right)+\sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)L_i\left(u\left(t_i\right)\right). \qquad (22)\end{aligned}$$

Then, the impulsive q_k-integrodifference equation (1) has a solution if and only if the operator equation $u = Qu$ has a fixed point.

In order to prove the existence of solutions for (1), we need the following known result (J. X. Sun, 2008).

Lemma 2. Let E be a Banach space. Assume that $T: E \to E$ is a completely continuous operator and the set $V = \{x \in E \,|\, x = \mu T x, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in E.

Theorem 3. Assume the following.

(H_1) There exist nonnegative bounded function $M_i(t)$ $(i = 1,2,3,4)$ such that $|f(t,u)| \le M_1(t) + M_2(t)|u|$, $|g(t,u)| \le M_3(t) + M_4(t)|u|$, for any $t \in J$, $u \in \mathbb{R}$,

denote $\sup_{t\in J}|M_i(t)| = M_i$, $i = 1,2,3,4$.

(H_2) There exist positive constants $\overline{L}, L'$ such that

$$|I_k(u)| \le \overline{L}, \quad |L_k(u)| \le L', \qquad (23)$$

for any $u \in \mathbb{R}$, $k = 1,2,\cdots,p$.

Then the problem (1) has at least one solution provided.

$$\tau = \sup_{t\in J}\left[\frac{M_2(t)}{2}\left(\sum_{i=0}^{p}\tau_{1i}+\frac{t^2-2t_kt-1+2t_k}{(1+q_k)}+p\right)+\frac{M_4(t)}{2}\left(\sum_{i=0}^{p}(1-q_i)\tau_{2i}+\sum_{i=0}^{p}\tau_{3i}+\frac{(1-q_k)\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k}+\frac{q_k^{\,2}\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k+q_k^{\,2}}+\sum_{i=0}^{p-1}\tau_{1i}\right)\right]<1$$

. (24)

Where $\tau_{1i} = (t_{i+1}-t_i)^2/1+q_i$, $\tau_{2i} = (t_{i+1}-t_i)^3/1+q_i$, $\tau_{3i} = q_i^2(t_{i+1}-t_i)^3/1+q_i+q_i^2$.

Proof. Firstly, we prove the operator $Q:PC^1(J,\mathbb{R})\to PC^1(J,\mathbb{R})$ is completely continuous. Clearly, continuity of the operator Q follows from the continuity of f,g,I_k. Let $\Omega\in PC^1(J,\mathbb{R})$ be bounded, then $\forall t\in J, u\in\Omega$. There exist positive constants $L_i\,(i=1,2,3,4)$ such that $|f(t,u)|\le L_1$, $|g(t,u)|\le L_2$, $|I_k(u)|\le L_3$, $|L_k(u)|\le L_4$. Thus

$$
\begin{aligned}
&|(Qu)(t)|\\
\le&\int_{t_k}^{t}[(t-t_k)-q_k(s-t_k)]\Big[|f(s,u(s))|+[(t-t_k)-q_k(s-t_k)]\times|g(s,u(s))|\Big]d_{q_k}s\\
&-\frac{1}{2}\int_{t_k}^{1}[(1-t_k)-q_k(s-t_k)]\Big[|f(s,u(s))|+[(1-t_k)-q_k(s-t_k)]\times|g(s,u(s))|\Big]d_{q_k}s\\
&+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}[(t_{i+1}-t_i)-q_i(s-t_i)]\Big[|f(s,u(s))|+\big[(t_{i+1}-t_i)-q_i(s-t_i)\big]\times|g(s,u(s))|\Big]d_{q_i}s\\
&+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\Big[|f(s,u(s))|+\big[(t_{i+1}-t_i)-q_i(s-t_i)\big]\times|g(s,u(s))|\Big]d_{q_i}s\\
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=1}^{p}|L_i(u(t_i))|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\Big[|f(s,u(s))|+\big[(t_{i+1}-t_i)-q_i(s-t_i)\big]\times|g(s,u(s))|\Big]d_{q_i}s\right)\\
&+\frac{1}{2}\sum_{i=1}^{k}|I_i(u(t_i))|+\sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)|L_i(u(t_i))|\\
\le&\int_{t_k}^{t}[(t-t_k)-q_k(s-t_k)][L_1+[(t-t_k)-q_k(s-t_k)]L_2]d_{q_k}s\\
&-\frac{1}{2}\int_{t_k}^{1}[(1-t_k)-q_k(s-t_k)][L_1+[(1-t_k)-q_k(s-t_k)]L_2]d_{q_k}s\\
&+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}[(t_{i+1}-t_i)-q_i(s-t_i)]\Big[L_1+\big[(t_{i+1}-t_i)-q_i(s-t_i)\big]L_2\Big]d_{q_i}s\\
&+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\Big[L_1+\big[(t_{i+1}-t_i)-q_i(s-t_i)\big]L_2\Big]d_{q_i}s\\
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=1}^{p}L_4+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\Big[L_1+\big[(t_{i+1}-t_i)-q_i(s-t_i)\big]L_2\Big]d_{q_i}s\right)\\
&+\frac{1}{2}\sum_{i=1}^{k}L_3+\sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)L_4\\
&\le(t-t_k)^2L_1+(t-t_k)^3L_2-\frac{2q_k(t-t_k)^3L_2}{1+q_k}-\frac{q_kL_1}{1+q_k}(t-t_k)^2+\frac{q_k{}^2L_2}{1+q_k+q_k^2}(t-t_k)^3\\
&-\frac{1}{2}\left[(1-t_k)^2L_1+(1-t_k)^3L_2-\frac{2q_k(1-t_k)^3L_2}{1+q_k}-\frac{q_kL_1}{1+q_k}(1-t_k)^2+\frac{q_k{}^2L_2}{1+q_k+q_k^2}(1-t_k)^3\right]\\
&+\frac{1}{2}\sum_{i=0}^{k-1}\Big[(t_{i+1}-t_i)^2L_1+(t_{i+1}-t_i)^3L_2-2q_i\tau_{2i}L_2-q_i\tau_{1i}L_1+\tau_{3i}L_2\Big]+\sum_{i=0}^{k-1}(t-\frac{1}{2}t_{i+1}-\frac{1}{2})\Big[(t_{i+1}-t_i)L_1+(t_{i+1}-t_i)^2L_2-q_i\tau_{1i}L_2\Big]\\
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left[pL_4+\sum_{i=0}^{p}[(t_{i+1}-t_i)L_1+(t_{i+1}-t_i)^2L_2-q_i\tau_{1i}L_2]\right]+\frac{p}{2}L_3+pL_4
\end{aligned}
$$

$$\le \frac{L_1}{2}\left(\sum_{i=0}^{p}\tau_{1i}+\frac{t^2-2t_kt-1+2t_k}{(1+q_k)}+p\right)+\frac{L_2}{2}\left(\sum_{i=0}^{p}(1-q_i)\tau_{2i}+\sum_{i=0}^{p}\tau_{3i}+\frac{(1-q_k)\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k}+\frac{q_k{}^2\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k+q_k{}^2}+\sum_{i=0}^{p-1}\tau_{1i}\right)$$

$$+\frac{p}{2}L_3+\frac{P}{2}L_4:=\varphi_1 \text{ (constant).} \quad (25)$$

This implies $\|Qu\|\le\varphi_1$.

And

$$\left|(D_{q_k}Qu)(t)\right|$$

$$\le\int_{t_k}^{t}\left[\left|f(s,u(s))\right|+[(t-t_k)-q_k(s-t_k)]\times\left|g(s,u(s))\right|\right]d_{q_k}s+\sum_{i=0}^{p}\left|L_i(u(t_i))\right|$$

$$+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left[\left|f(s,u(s))\right|+[(t_{i+1}-t_i)-q_i(s-t_i)]\left|g(s,u(s))\right|\right]d_{q_i}s$$

$$-\frac{1}{2}\left(\sum_{i=1}^{p}\left|L_i(u(t_i))\right|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[\left|f(s,\mathrm{u}(s))\right|+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]\times\left|g(s,u(s))\right|\right]d_{q_i}s\right)$$

$$\le\int_{t_k}^{t}[L_1+[(t-t_k)-q_k(s-t_k)]L_2]d_{q_k}s+\sum_{i=1}^{k}L_4+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}[L_1+[(t_{i+1}-t_i)-q_i(s-t_i)]L_2]d_{q_i}s$$

$$-\frac{1}{2}\left(\sum_{i=1}^{p}L_4+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[L_2+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]L_2\right]d_{q_i}s\right)$$

$$\le(t-t_k)L_1+(t-t_k)^2L_2-\frac{q_k(t-t_k)^2}{1+q_k}L_2+pL_4+\sum_{i=0}^{k-1}\left[(t_{i+1}-t_i)L_1+(t_{i+1}-t_i)^2L_2-q_i\tau_{1i}L_2\right]$$

$$-\frac{1}{2}\left(pL_4+\sum_{i=0}^{p}[(t_{i+1}-t_i)L_1+(t_{i+1}-t_i)^2L_2-q_i\tau_{1i}L_2]\right)$$

$$\le\frac{p}{2}L_4+\frac{L_1}{2}\sum_{i=0}^{p}(t_{i+1}-t_i)+\frac{1}{2}\sum_{i=0}^{p}\tau_{1i}L_2:=\varphi_2 \text{ (constant).} \quad (26)$$

This implies $\|D_{q_k}Qu\|\le\varphi_2$.

Furthermore, for any $t',t''\in J_k\,(k=0,1,2,\ldots,p)$ satisfying $t'<t''$, we have

$$\left|(Qu)(t'')-(Qu)(t')\right|$$

$$\le\int_{t_k}^{t''}[(t''-t_k)-q_k(s-t_k)]\left[\left|f(s,u(s))\right|+[(t''-t_k)-q_k(s-t_k)]\left|g(s,u(s))\right|\right]d_{q_k}s$$

$$+\sum_{i=0}^{k-1}\left(t''-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\left[\left|f(s,u(s))\right|+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]\times\left|g(s,u(s))\right|\right]d_{q_i}s$$

$$+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t''\right)\left(\sum_{i=1}^{p}\left|L_i(u(t_i))\right|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[\left|f(s,\mathrm{u}(s))\right|+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]\times\left|g(s,u(s))\right|\right]d_{q_i}s\right)$$

$$-\int_{t_k}^{t'}[(t'-t_k)-q_k(s-t_k)]\left[\left|f(s,u(s))\right|+[(t'-t_k)-q_k(s-t_k)]\left|g(s,u(s))\right|\right]d_{q_k}s$$

$$-\sum_{i=0}^{k-1}\left(t'-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\left[\left|f(s,u(s))\right|+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]\times\left|g(s,u(s))\right|\right]d_{q_i}s$$

$$-\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t'\right)\left(\sum_{i=1}^{p}\left|L_i\left(u\left(t_i\right)\right)\right|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[\left|f\left(s,\mathrm{u}\left(s\right)\right)\right|+\left[\left(t_{i+1}-t_i\right)-q_i\left(s-t_i\right)\right]\times\left|g\left(s,u\left(s\right)\right)\right|\right]d_{q_i}s\right)$$

$$+\sum_{i=1}^{k}\left(t''-\frac{1}{2}t_i-\frac{1}{2}\right)\left|L_i\left(u\left(t_i\right)\right)\right|-\sum_{i=1}^{k}\left(t'-\frac{1}{2}t_i-\frac{1}{2}\right)\left|L_i\left(u\left(t_i\right)\right)\right|$$

$$\le\int_{t'}^{t''}\left[(t''-t_k)-q_k(s-t_k)\right]\left[\left|f\left(s,u(\mathrm{s})\right)\right|+[(t''-t_k)-q_k(s-t_k)]\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s$$
$$+\int_{t_k}^{t'}\left[(t''-t')\left|f\left(s,u(\mathrm{s})\right)\right|+[(t''^2-t'^2)-2t_k(t''-t')-2q_k(s-t_k)(t''-t')]\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s$$
$$+(t''-t')\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left[\left|f\left(s,u(\mathrm{s})\right)\right|+\left[\left(t_{i+1}-t_i\right)-q_i\left(s-t_i\right)\right]\times\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_i}s+\sum_{i=1}^{k}(t''-t')\left|L_i\left(u(t_i)\right)\right|$$
$$-\frac{(t''-t')}{2}\left(\sum_{i=1}^{p}\left|L_i\left(u\left(t_i\right)\right)\right|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[\left|f\left(s,\mathrm{u}\left(s\right)\right)\right|+\left[\left(t_{i+1}-t_i\right)-q_i\left(s-t_i\right)\right]\times\left|g\left(s,u\left(s\right)\right)\right|\right]d_{q_i}s\right)$$
$$\le\int_{t'}^{t''}\left[(t''-t_k)\left|f\left(s,u(\mathrm{s})\right)\right|+(t''-t_k)\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s$$
$$+\int_{t_k}^{t'}\left[(t''-t')\left|f\left(s,u(\mathrm{s})\right)\right|+[(t''^2-t'^2)-2t_k(t''-t')]\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s$$
$$+(t''-t')\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left[\left|f\left(s,u(\mathrm{s})\right)\right|+\left(t_{i+1}-t_i\right)\times\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_i}s$$
$$-\frac{(t''-t')}{2}\left(\sum_{i=1}^{p}\left|L_i\left(u\left(t_i^-\right)\right)\right|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[\left|f\left(s,\mathrm{u}\left(s\right)\right)\right|+\left(t_{i+1}-t_i\right)\left|g\left(s,u\left(s\right)\right)\right|\right]d_{q_i}s\right)+(t''-t')pL_4$$
$$\le[L_1+(t''-t_k)^2L_2+\frac{p}{2L_4}](t''-t')+[(t''+t')-2t_k]L_2(t'-t_k)(t''-t')$$
$$+\sum_{i=0}^{p-1}\left[\left(t_{i+1}-t_i\right)L_1+\left(t_{i+1}-t_i\right)^2L_2\right]\left[1-\frac{1}{2}\left(L_1+L_2\right)\right]\left(t''-t'\right). \qquad (27)$$

And

$$\left|(D_{q_k}Qu)(t'')-(D_{q_k}Qu)(t')\right|$$
$$\le\int_{t_k}^{t''}\left[\left|f\left(s,u(\mathrm{s})\right)\right|+[(t''-t_k)-q_k(s-t_k)]\times\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s$$
$$-\int_{t_k}^{t'}\left[\left|f\left(s,u(\mathrm{s})\right)\right|+[(t'-t_k)-q_k(s-t_k)]\times\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s$$
$$\le\int_{t'}^{t''}\left[\left|f\left(s,u(\mathrm{s})\right)\right|+[(t''-t_k)-q_k(s-t_k)]\times\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s+\int_{t_k}^{t'}(t''-t')\left|g\left(s,u(\mathrm{s})\right)\right|d_{q_k}s$$
$$\le\int_{t'}^{t''}\left[\left|f\left(s,u(\mathrm{s})\right)\right|+(t''-t_k)\times\left|g\left(s,u(\mathrm{s})\right)\right|\right]d_{q_k}s+L_2(t'-t_k)(t''-t')$$
$$\le[L_1+L_2(t''-t_k)](t''-t')+L_2(t'-t_k)(t''-t'). \qquad (28)$$

As $t'\to t''$, the right hand side of the above inequality tends to zero. Thus, $Q(\Omega)$ is relatively compact. As a consequence of Arzela Ascoli's theorem, Q is a compact operator. Therefore, Q is a completely continuous operator.

Define the set $W_1=\{u\in PC^1(J,\mathbb{R})\,|\,u=\lambda Qu,0<\lambda<1\}$. Next, we show W_1 is bounded. Let $u\in W_1$; then $u=\lambda Qu$, $0<\lambda<1$. For any $t\in J$ by conditions (H_1) and (H_2), we have

$$
\begin{aligned}
&|u(t)| = \lambda |(Qu)(t)| \\
&\le \int_{t_k}^{t} [(t-t_k)-q_k(s-t_k)]\Big[\big|f(s,u(s))\big| + [(t-t_k)-q_k(s-t_k)]\times\big|g(s,u(s))\big|\Big] d_{q_k}s \\
&-\frac{1}{2}\int_{t_k}^{1} [(1-t_k)-q_k(s-t_k)]\Big[\big|f(s,u(s))\big| + [(1-t_k)-q_k(s-t_k)]\times\big|g(s,u(s))\big|\Big] d_{q_k}s \\
&+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} [(t_{i+1}-t_i)-q_i(s-t_i)]\Big[\big|f(s,u(s))\big| + \big[(t_{i+1}-t_i)-q_i(s-t_i)\big]\times\big|g(s,u(s))\big|\Big] d_{q_i}s \\
&+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\Big[\big|f(s,u(s))\big| + \big[(t_{i+1}-t_i)-q_i(s-t_i)\big]\times\big|g(s,u(s))\big|\Big] d_{q_i}s \\
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\Big[\big|f(s,u(s))\big| + \big[(t_{i+1}-t_i)-q_i(s-t_i)\big]\times\big|g(s,u(s))\big|\Big] d_{q_i}s\right) \\
&+\frac{1}{2}\sum_{i=1}^{k}\big|I_i(u(t_i))\big| + \sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)\big|L_i(u(t_i))\big| + \left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\sum_{i=1}^{p}\big|L_i(u(t_i))\big| \\
&\le \int_{t_k}^{t} [(t-t_k)-q_k(s-t_k)]\Big[M_1+M_2|u(s)| + [(t-t_k)-q_k(s-t_k)](M_3+M_4|u(s)|)\Big] d_{q_k}s \\
&-\frac{1}{2}\int_{t_k}^{1} [(1-t_k)-q_k(s-t_k)]\Big[M_1+M_2|u(s)| + [(1-t_k)-q_k(s-t_k)](M_3+M_4|u(s)|)\Big] d_{q_k}s \\
&+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}} [(t_{i+1}-t_i)-q_i(s-t_i)]\Big[M_1+M_2|u(s)| + \big[(t_{i+1}-t_i)-q_i(s-t_i)\big](M_3+M_4|u(s)|)\Big] d_{q_i}s \\
&+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\Big[M_1+M_2|u(s)| + \big[(t_{i+1}-t_i)-q_i(s-t_i)\big](M_3+M_4|u(s)|)\Big] d_{q_i}s \\
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\Big[M_1+M_2|u(s)| + \big[(t_{i+1}-t_i)-q_i(s-t_i)\big](M_3+M_4|u(s)|)\Big] d_{q_i}s\right) \\
&+\frac{1}{2}\sum_{i=1}^{k}\bar{L} + \sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)L' + \left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=1}^{p}L'\right) \\
&\le (t-t_k)^2(M_1+M_2\|u\|) + (t-t_k)^3(M_3+M_4\|u\|) - \frac{2q_k(t-t_k)^3}{1+q_k}(M_3+M_4\|u\|) \\
&-\frac{q_k(t-t_k)^2}{1+q_k}(M_1+M_2\|u\|) + \frac{q_k^{\,2}(t-t_k)^3}{1+q_k+q_k^2}(M_3+M_4\|u\|) + \frac{1}{2}\sum_{i=1}^{k}\bar{L} + \frac{1}{2}\sum_{i=1}^{P}L' \\
&-\frac{1}{2}\left[(1-t_k)^2(M_1+M_2\|u\|) + (1-t_k)^3(M_3+M_4\|u\|) - \frac{2q_k(1-t_k)^3}{1+q_k}(M_3+M_4\|u\|) - \frac{q_k(1-t_k)^2}{1+q_k}(M_1+M_2\|u\|) + \frac{q_k^{\,2}(1-t_k)^3}{1+q_k+q_k^2}(M_3+M_4\|u\|)\right] \\
&+\frac{1}{2}\sum_{i=0}^{k-1}\Big[(t_{i+1}-t_i)^2(M_1+M_2\|u\|) + (t_{i+1}-t_i)^3(M_3+M_4\|u\|) - 2q_i\tau_{2i}(M_3+M_4\|u\|) - q_i\tau_{1i}(M_1+M_2\|u\|) + \tau_{3i}(M_3+M_4\|u\|)\Big] \\
&+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\Big[(t_{i+1}-t_i)(M_1+M_2\|u\|) + (t_{i+1}-t_i)^2(M_3+M_4\|u\|) - q_i\tau_{1i}(M_3+M_4\|u\|)\Big] \\
&-\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=0}^{p}\Big[(t_{i+1}-t_i)(M_1+M_2\|u\|) + (t_{i+1}-t_i)^2(M_3+M_4\|u\|) - q_i\tau_{1i}(M_3+M_4\|u\|)\Big]\right) \\
&+\frac{1}{2}\sum_{i=1}^{p}L' + \frac{1}{2}\sum_{i=1}^{k}\bar{L}
\end{aligned}
$$

$$\leq \frac{M_1}{2}\left(\sum_{i=0}^{p}\tau_{1i}+\frac{t^2-2t_kt-1+2t_k}{(1+q_k)}+p\right)+\frac{M_3}{2}\left(\sum_{i=0}^{p}(1-q_i)\tau_{2i}+\sum_{i=0}^{p}\tau_{3i}+\frac{(1-q_k)\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k}+\frac{q_k^{\,2}\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k+q_k^{\,2}}+\sum_{i=0}^{p-1}\tau_{1i}\right)$$

$$+\|u\|\left(\frac{M_2}{2}\left(\sum_{i=0}^{p}\tau_{1i}+\frac{t^2-2t_kt-1+2t_k}{(1+q_k)}+p\right)+\frac{M_4}{2}\left(\sum_{i=0}^{p}(1-q_i)\tau_{2i}+\sum_{i=0}^{p}\tau_{3i}+\frac{(1-q_k)\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k}+\frac{q_k^{\,2}\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k+q_k^{\,2}}+\sum_{i=0}^{p-1}\tau_{1i}\right)\right)$$

$$+\frac{p}{2}L'+\frac{p}{2}\bar{L}. \qquad (29)$$

$$\|u\|\leq\frac{1}{1-\tau}\left(\frac{M_1}{2}\left(\sum_{i=0}^{p}\tau_{1i}+\frac{t^2-2t_kt-1+2t_k}{(1+q_k)}+p\right)+\frac{M_3}{2}\left(\sum_{i=0}^{p}(1-q_i)\tau_{2i}+\sum_{i=0}^{p}\tau_{3i}+\frac{(1-q_k)\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k}+\frac{q_k^{\,2}\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k+q_k^{\,2}}+\sum_{i=0}^{p-1}\tau_{1i}\right)\right)$$

. (30) Similarly, for any $t\in J$ by conditions (H_1) and (H_2), we have

$$\left|\left(D_{q_k}u\right)(t)\right|=\lambda\left|\left(D_{q_k}u\right)(t)\right|$$

$$\leq\int_{t_k}^{t}\left[\left|f(s,u(s))\right|+[(t-t_k)-q_k(s-t_k)]\times\left|g(s,u(s))\right|\right]d_{q_k}s+\sum_{i=1}^{p}\left|L_i(u(t_i))\right|$$

$$+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left[\left|f(s,u(s))\right|+[(t_{i+1}-t_i)-q_i(s-t_i)]\left|g(s,u(s))\right|\right]d_{q_i}s$$

$$-\frac{1}{2}\left(\sum_{i=1}^{p}\left|L_i(u(t_i))\right|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[\left|f(s,u(s))\right|+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]\times\left|g(s,u(s))\right|\right]d_{q_i}s\right)$$

$$\leq\int_{t_k}^{t}\left[M_1+M_2|u(s)|+[(t-t_k)-q_k(s-t_k)](M_3+M_4|u(s)|)\right]d_{q_k}s+\sum_{i=1}^{k}L'-\frac{1}{2}\sum_{i=1}^{p}L'$$

$$+\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}\left[M_1+M_2|u(s)|+[(t_{i+1}-t_i)-q_i(s-t_i)](M_3+M_4|u(s)|)\right]d_{q_i}s$$

$$-\frac{1}{2}\left(\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[M_1+M_2|u(s)|+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]\left(M_3+M_4|u(s)|\right)\right]d_{q_i}s\right)$$

$$\leq(t-t_k)(M_1+M_2\|u\|)+(t-t_k)^2(M_3+M_4\|u\|)-\frac{q_k(t-t_k)^2}{1+q_k}(M_3+M_4\|u\|)$$

$$+\sum_{i=0}^{k-1}\left[(t_{i+1}-t_i)(M_1+M_2\|u\|)+(t_{i+1}-t_i)^2(M_3+M_4\|u\|)-q_i\tau_{1i}(M_3+M_4\|u\|)\right]$$

$$-\frac{1}{2}\left(\sum_{i=0}^{p}[(t_{i+1}-t_i)(M_1+M_2\|u\|)+(t_{i+1}-t_i)^2(M_3+M_4\|u\|)-q_i\tau_{1i}(M_3+M_4\|u\|)]\right)+\frac{p}{2}L'$$

$$\leq\frac{p}{2}L'+\frac{M_1}{2}\sum_{i=0}^{p}(t_{i+1}-t_i)+\frac{1}{2}\sum_{i=0}^{p}\tau_{1i}M_3+\|u\|\left[\frac{M_2}{2}\sum_{i=0}^{p}(t_{i+1}-t_i)+\frac{1}{2}\sum_{i=0}^{p}\tau_{1i}M_4\right].$$

(31)

$$\left\|D_{q_k}u\right\|\leq\frac{1}{1-\tau'}\left(pL'+M_1\sum_{i=0}^{p}(t_{i+1}-t_i)+M_3\sum_{I=0}^{P}\tau_{1i}\right):=\text{constant}. \qquad (32)$$

Where $\tau'=\left[M_2\sum_{i=0}^{p}(t_{i+1}-t_i)+M_4\sum_{I=0}^{P}\tau_{1i}\right]$, $\|u\|_{PC^1}=\sup_{t\in J}\left\{\|u\|_{PC},\|D_{qk}u\|_{PC}\right\}$,

$\|u\|_{PC^1}\le\tau$.

So, the set W_1 is bounded. Thus, Lemma 2 ensures the impulsive q_k - integrodifference equation (1) has at least one solution.

Corollary 4. Assume the following.

(H_3) There exist nonnegative constants $L_i\left(i=1,2,3,4\right)$ such that

$$|f(t,u)|\le L_1,\ |g(t,u)|\le L_2,\ |I_k(u)|\le L_3,\ |L_k(u)|\le L_4, \qquad (33)$$

for any $t\in J$, $u\in\mathbb{R}$, $k=1,2,\ldots p$. Then problem (1) has at least one solution.

Theorem 5. Assume the following.

(H_4) There exist nonnegative bounded functions $M(t)$ and $N(t)$ such that

$$|f(t,u)-f(t,v)|\le M(t)|u-v|,\ |g(t,u)-g(t,v)|\le N(t)|u-v|, \qquad (34)$$

for $t\in J$, $u,v\in\mathbb{R}$.

(H_5) There exist positive constants K, G, X such that

$$|I_k(u)-I_k(v)|\le K|u-v|,\ |L_k(u)-L_k(v)|\le X|u-v|, \qquad (35)$$

for $u,v\in\mathbb{R}$ and $k=1,2,\ldots,p$.

(H_6)

$$K_1=\sup_{t\in J}\left[\frac{M_1}{2}\left(\sum_{i=0}^{p}\tau_{1i}+\frac{t^2-2t_kt-1+2t_k}{(1+q_k)}+p\right)+\frac{M_3}{2}\left(\sum_{i=0}^{p}(1-q_i)\tau_{2i}+\sum_{i=0}^{p}\tau_{3i}+\frac{(1-q_k)\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k}+\frac{q_k{}^2\left[(t-t_k)^3-(1-t_k)^3\right]}{1+q_k+q_k{}^2}+\sum_{i=0}^{p-1}\tau_{1i}\right)+\frac{P}{2}X+\frac{P}{2}K\right]<1$$

. (36)

Then problem (1) has a unique solution.

Proof. Clearly Q is a continuous operator. Denote $\sup_{t\in J}|M(t)|=M$, $\sup_{t\in J}|N(t)|=N$. For $\forall u,v\in PC^1(J,\mathbb{R})$, by ($H_4$) and ($H_5$), we have

$$|(Qu)(t)-(Qv)(t)|$$

$$\le\int_{t_k}^{t}[(t-t_k)-q_k(s-t_k)]\left[\left|f(s,u(s))-f(s,v(s))\right|+[(t-t_k)-q_k(s-t_k)]\left|g(s,u(s))-g(s,v(s))\right|\right]d_{q_k}s$$

$$-\frac{1}{2}\int_{t_k}^{1}[(1-t_k)-q_k(s-t_k)]\left[\left|f(s,u(s))-f(s,v(s))\right|+[(1-t_k)-q_k(s-t_k)]\left|g(s,u(s))-g(s,v(s))\right|\right]d_{q_k}s$$

$$+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}[(t_{i+1}-t_i)-q_i(s-t_k)]\left[\left|f(s,u(s))-f(s,v(s))\right|+[(t_{i+1}-t_i)-q_i(s-t_i)]\left|g(s,u(s))-g(s,v(s))\right|\right]d_{q_i}s$$

$$+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\left[\left|f(s,u(s))-f(s,v(s))\right|+\left[(t_{i+1}-t_i)-q_i(s-t_i)\right]\times\left|g(s,u(s))-g(s,v(s))\right|\right]d_{q_i}s$$

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$$
\begin{aligned}
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=1}^{p}\left|L_i\left(u\left(t_i\right)\right)-L_i\left(v\left(t_i\right)\right)\right|+\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[\left|f\left(s,u\left(s\right)\right)-f\left(s,v\left(s\right)\right)\right|+\left[\left(t_{i+1}-t_i\right)-q_i\left(s-t_i\right)\right]\times\left|g\left(s,u\left(s\right)\right)-g\left(s,v\left(s\right)\right)\right|\right]d_{q_i}s\right)\\
&+\frac{1}{2}\sum_{i=1}^{k}\left|I_i\left(u\left(t_i\right)\right)-I_i\left(v\left(t_i\right)\right)\right|+\sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)\left|L_i\left(u\left(t_i\right)\right)-L_i\left(v\left(t_i\right)\right)\right|\\
&\le\int_{t_k}^{t}\left[(t-t_k)-q_k(s-t_k)\right]\left[M(s)+[(t-t_k)-q_k(s-t_k)]N(s)\right]\left|(u-v)(s)\right|d_{q_k}s\\
&-\frac{1}{2}\int_{t_k}^{1}\left[(1-t_k)-q_k(s-t_k)\right]\left[M\left(s\right)+[(1-t_k)-q_k(s-t_k)]N\left(s\right)\right]\left|(u-v)\right|d_{q_k}s\\
&+\frac{1}{2}\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}[(t_{i+1}-t_i)-q_i(s-t_i)]\left[M\left(s\right)+\left[\left(t_{i+1}-t_i\right)-q_i\left(s-t_i\right)\right]N\left(s\right)\right]\left|(u-v)\right|d_{q_i}s\\
&+\sum_{i=0}^{k-1}\left(t-\frac{1}{2}t_{i+1}-\frac{1}{2}\right)\int_{t_i}^{t_{i+1}}\left[M\left(s\right)+\left[\left(t_{i+1}-t_i\right)-q_i\left(s-t_i\right)\right]N\left(s\right)\right]\left|(u-v)\right|d_{q_i}s\\
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=0}^{p}\int_{t_i}^{t_{i+1}}\left[M\left(s\right)+\left[\left(t_{i+1}-t_i\right)-q_i\left(s-t_i\right)\right]N\left(s\right)\right]\left|u-v\right|d_{q_i}s\right)\\
&+\frac{1}{2}\sum_{i=1}^{k}K\left|(u-v)(t_i)\right|+\sum_{i=1}^{k}\left(t-\frac{1}{2}t_i-\frac{1}{2}\right)X\left|(u-v)(t_i)\right|+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\sum_{i=1}^{p}X\left|(u-v)(t_i)\right|\\
&\le\left[(t-t_k)^2M+(t-t_k)^3N-\frac{2q_k(t-t_k)^3}{1+q_k}N-\frac{q_k(t-t_k)^2}{1+q_k}M+\frac{q_k{}^2(t-t_k)^3}{1+q_k+q_k^2}N\right]\|u-v\|\\
&-\frac{1}{2}\left[(1-t_k)^2M+(1-t_k)^3N-\frac{2q_k(1-t_k)^3}{1+q_k}N-\frac{q_k(1-t_k)^2}{1+q_k}M+\frac{q_k{}^2(1-t_k)^3}{1+q_k+q_k^2}N\right]\|u-v\|\\
&+\frac{1}{2}\sum_{i=0}^{k-1}\left[(t_{i+1}-t_i)^2M+(t_{i+1}-t_i)^3N-2q_i\tau_{2i}N-q_i\tau_{1i}M+\tau_{3i}N\right]\|u-v\|\\
&+\sum_{i=0}^{k-1}(t-\frac{1}{2}t_{i+1}-\frac{1}{2})\left[(t_{i+1}-t_i)M+(t_{i+1}-t_i)^2N-q_i\tau_{1i}N\right]\|u-v\|\\
&+\left(\frac{1}{4}+\frac{1}{4}t_0-\frac{1}{2}t\right)\left(\sum_{i=0}^{p}[(t_{i+1}-t_i)M+(t_{i+1}-t_i)^2N-q_i\tau_{1i}N]\|u-v\|\right)+\left(\frac{p}{2}X+\frac{p}{2}K\right)\|u-v\|\\
&\le\left[\frac{M}{2}\left(\sum_{i=0}^{p}\tau_{1i}+\frac{t^2-2t_kt-1+2t_k}{(1+q_k)}+p\right)+\frac{N}{2}\left(\sum_{i=0}^{p}(1-q_i)\tau_{2i}+\sum_{i=0}^{p}\tau_{3i}+\frac{(1-q_k)\left[\left(t-t_k\right)^3-\left(1-t_k\right)^3\right]}{1+q_k}+\frac{q_k{}^2\left[\left(t-t_k\right)^3-\left(1-t_k\right)^3\right]}{1+q_k+q_k{}^2}+\sum_{i=0}^{p-1}\tau_{1i}\right)+\left(\frac{p}{2}X+\frac{p}{2}K\right)\right]\|u-v\|\\
&\le K_1\|u-v\|. \qquad (37)
\end{aligned}
$$

As $K_1<1$ by (H_6). Therefore, Q is a contractive map. Thus, the conclusion of the Theorem 5 follows by Banach contraction mapping principle.

EXAMPLE

Consider the following second order anti-periodic nonlinear q_k- integrodifference equation with impulses

$$
\begin{aligned}
D^2_{1/(2+k)}u(t)&=8+3\sqrt{t}+\ln\left(1+5t^3+\frac{t^2}{5}\left|u(t)\right|\right)\\
&+\int_{1/(1+2k)}^{t}\left[10s+\frac{s^3}{3}\sin u(s)\right]d_{1/(2+k)}s,\quad t\in(0,1),\ t\neq\frac{1}{1+2k},
\end{aligned}
$$

$$D_{1/(2+k)}u\left(\frac{1}{1+2k}\right)-D_{1/(1+k)}u\left(\frac{1}{1+2k}\right)=\sin\left(u\left(\frac{1}{1+2k}\right)\right),$$

$$\Delta u\left(\frac{1}{1+2k}\right)=\cos\left(u\left(\frac{1}{1+2k}\right)\right), \qquad k=1,2,\cdots,6,$$

$$u(0)=-u(1^-),\ D_{1/2}u(0^+)=-D_{1/9}u(1^-). \qquad (38)$$

Obviously, $q_k=1/(2+k)(k=0,1,2,\cdots,6)$, $t_k=1/(1+2k)(k=1,2,\cdots,6)$, $f(t,u)=8+3\sqrt{t}+\ln(1+5t^3+\frac{t^2}{5}|u(t)|)$, $g(t,u)=10t+\frac{t^3}{3}\sin u$, $I_k(u)=\cos u$, $L_k(u)=\sin u$.

By a simple calculation, we can get

$$|f(t,u)|\le 8+3\sqrt{t}+5t^3+\frac{t^2}{5}|u(t)|, \quad |g(t,u)|\le 10t+\frac{t^3}{3}|u|, \qquad (39)$$

$$|I_k(u)|\le 1, \quad |L_k(u)|\le 1.$$

Take $M_1(t)=8+3\sqrt{t}+5t^3$, $M_2(t)=\frac{t^2}{5}$, $M_3(t)=10t$, $M_4(t)=\frac{t^3}{3}$, and $\bar{L}=L'=1$. Then all conditions of Theorem 3 hold. By Theorem 3, second order anti-periodic nonlinear impulsive q_k-integrodifference (38) has at least one solution.

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