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Local textures parameters of images in space domain obtained with different analytical approaches

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Abstract

The aim of this study is to determine and compare, in the spatial domain the different local texture parameters of 2D image. For this, we define the realizations of analytical images obtained in the spatial domain with three transformation approaches (Fourier, Hamilton and Riesz). The generalization to 3D image is envisaged.

Keywords: Textures, analytical signal, Fourier, Hamilton, Riesz, Transformations

I- Introduction

In signal processing, the substitution of a real signal by the corresponding complex analytic signals is widely applied. Among many advantages, it is possible to define different local characteristic, generalization of the concept of instantaneous parameter, frequently used in 1D temporal signal processing.

Different Transformations can represent a real 2D image in complex form. Besides the classic 2D Fourier Transform, which gives access to a half spectral plane and after inverse transformation to an analytical Image As, we can quote, in particular, the Hamilton (quaternionic) and the Riesz (monogenic) transforms. These transformations lead us, respectively, to the representations in the spatial plane of 2D analytic signal As, QS quaternion signal and the monogenic signal MS .These different tools, whose properties are partially complementary, can restore the original image, to avoid spatial redundancies and extend the concepts in the space domain to amplitude, phase and orientation.

II- 2D Complex signals

This part introduces the 2D complex Fourier analytical signal, the quaternionic signal and the monogenic signal.

II -1 2D Analytic Fourier Signal

The original 2D real image B/W is $u(x_1,x_2)$.The classical Fourier Transform is:

$$U(f_1,f_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-j2\pi x1f1} u(x1x2) e^{-j2\pi x2f2} dx1dx2$$

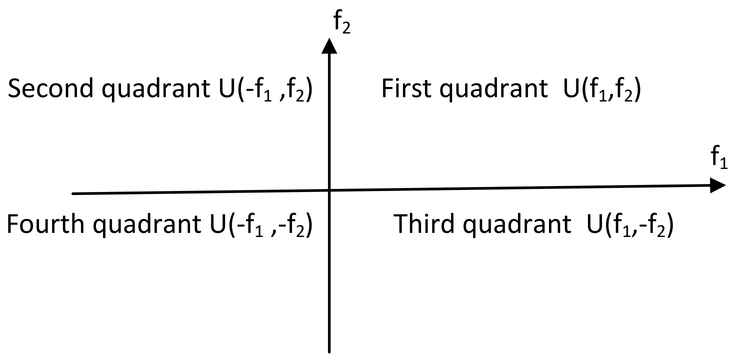

Fig 1: The four-quadrant Fourier spectrum of 2D real signal

(in frequency domain)

An important feature of the 2DFT is the Hermitian symmetry of this spectrum:

$U(f_1,f_2)=U^*(-f_1-f_2)$ $\quad *$ conjugate expression

$U(-f_1,f_2)=U^*(f_1-f_2)$

Due to this symmetry, the signal can be recovered from its half plane spectrum

(Quadrant 1, with $f_1>0$ $f_2>0$ and Quadrant 3 ,with $f_1>0$ $f_2<0$.for example).

2D complex signals with single-quadrant spectra

Two-dimensional complex signals with single-quadrant spectra were proposed by Hahn (Ref 1,5,6) .This signal can be obtained by 2D complex Fourier transformation by taking one quadrant's information in its frequency spectrum

The half-plane spectrum of the first quadrant and the third quadrant in Fig1 is chosen. The single-quadrant $2DA_S$ from the first and the third quadrant of the spectrum can be defined as $\Psi_1(x_1,x_2)$ and $\Psi_2(x_1,x_2)$ respectively

Segmentation of the spectrum:

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The analytical signal A_s is the inverse Fourier Transform of the spectrum in this first quadrant is $\Psi_1(x_1,x_2)$. Similarly, the A_S with the third quadrant is: $\Psi_2(x_1,x_2)$

As_1= $\Psi_1(x_1,x_2)= F^{-1}\{[1+sgn(f_1)][1+sgn(f_2)]\}\ U(f_1,f_2)$ $\qquad \Psi_1(x_1,x_2)=u-\nu+j(\nu_1+\nu_2)$

As_2= $\Psi_2(x_1,x_2)= F^{-1}\{[1+sgn(f_1)][1-sgn(f_2)]\}\ U(f_1,f_2)$ $\qquad \Psi_2(x_1,x_2)=u+\nu+j(\nu_1-\nu_2)$

$$\nu(x_1,x_2)=u(x_1,x_2)**\frac{1}{\pi^2 x_1 x_2}\ ;\ \nu_1(x_1,x_2)= u(x_1,x_2)*\frac{\delta(x_2)}{\pi x_1}\ ;\ \nu_2(x_1,x_2)=u(x_1,x_2)*\frac{\delta(x_1)}{\pi x_2}$$

We obtain, in Polar notation:

$$\Psi_1(x_1,x_2)= A_1(x_1,x_2)\ e^{j\phi_1(x_1,x_2)}$$

$$\Psi_2(x_1,x_2)= A_2(x_1,x_2)\ e^{j\phi_2(x_1,x_2)}$$

where the local amplitudes are given by the relations:

$$A_1(x_1,x_2) = \sqrt{u^2+\nu^2+\nu_1^2+\nu_2^2+2(\nu_1\nu_2\ -u\nu)}$$

$$A_2(x_1,x_2) = \sqrt{u^2+\nu^2+\nu_1^2+\nu_2^2-2(\nu_1\nu_2\ -u\nu)}$$

and the local phase functions are:

$$\phi_1 = Arctg(\frac{\nu_1+\nu_2}{u-\nu})$$

$$\phi_2 = Arctg(\frac{\nu_1-\nu_2}{u-\nu})$$

A real 2D signal $u(x_1,x_2)$ can be represented by two local Amplitude A_1 ,A_2 and two phase functions ϕ_1 and ϕ_2

II-2 Quaternionic Fourier Signal (Hamilton)

Quaternions , uncovered in 1843 by W.R. Hamilton ,is a generalized complex number of the form w+xi+yj+zk, where w,x,y,z are real numbers and i,j,k imaginaire units that satisfy the conditions:

$$i^2 = j^2 = k^2=-1 \text{ and } ij = -ji =k\ ,\ jk = -ki =i \text{ and } ki = -ik = j$$

This Quaternionic Fourier Transform QFT(Ref 2 ,7) has many different types. In this presentation, we only use the type 1, which has the following expression: :

$$Uq\ (f1,f2)\ = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-i2\pi x1f1}u(x1,x2)e^{-j2\pi x2f2}dx1dx2$$

if the 2D real signal B/W is $u(x_1,x_2)$.

This spectrum has four quadrants .It is a quaternion, obeys the rules of the quaternion Hermitian symmetry ,defined by the relations:

$$\alpha_i\,[U_q(f_1,f_2)] = U_q(f_1,-f_2)$$

$$\alpha_j[\,U_q(f_1,f_2)] = U_q(-f_1,f_2)$$

$$\alpha_k[U_q(f_1,f_2)] = U_q(-f_1,-f_2)$$

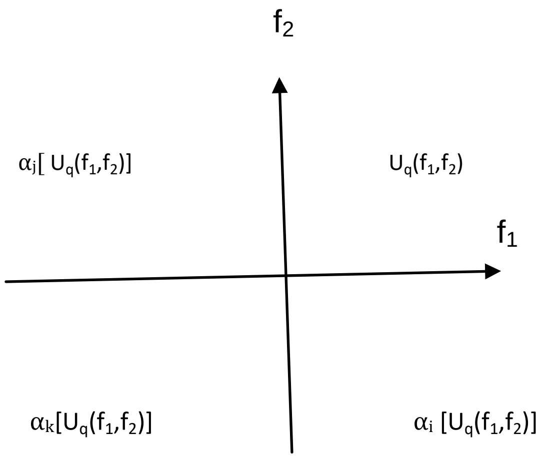

Fig 2 : Involutions : $\propto_i$,$\propto_j$ and $\propto_k$ are the involutions of U_q

Segmentation of spectrum QFT :

The spectrum mono quadrant is $\Gamma_q(f_1,f_2)$

$$\Gamma_q(f_1,f_2) = [1+\mathrm{sgn}(f_1)][1+\mathrm{sgn}(f_2)]\,U_q(f_1,f_2)$$

Any real signal can be reconstructed using only a single quadrant of the spectrum QFT (Ref 5,6).

Quaternionic signal $\Psi_q\,(x1,x2)$

$$\Psi_q\,(x1,x2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{+i2\pi x1 f1}\,\Gamma q\,(f1,f2)\; e^{+j2\pi x2 f2}\; df_1 df_2$$

The integration yields :

$$\Psi_q\,(x1,x2) = u\,(x1,x2) + i\nu_1(x1,x2) + j\nu_2(x1,x2) + k\nu\,(x1,x2)$$ where ν_1 ν_2 and ν are total and partial Hilbert Transforms .

The real part of this quaternionic signal is also the original signal $u(x1,x2)$

The polar representation define a single local amplitude A_q and three phase functions Φi , Φ_j and Φ_k

$$\Psi_q\ (x1, x2) = A_q\, e^{\Phi i}\, e^{\Phi j}\, e^{\Phi k}$$

The amplitude is Aq $(x_1,x_2)= \sqrt{u^2+v^2+v_1^2+v_2^2}$

The phase angles are:

$$\Phi_i = 0.5 \mathrm{Arctg}\ \ 2(u\, v_1 + v\, v_2) / \ u^2 - v^2 - v_1^2 + v_2^2$$

$$\Phi_J = 0.5 \mathrm{Arctg}\ \ 2(u\, v_2 + v\, v_1) / \ u^2 - v^2 + v_1^2 - v_2^2$$

$$\Phi_k = 0.5 \mathrm{Arctg}\ \ 2(u\, v + v_1 v_2) / \ A_q^2$$

The integration yields :

$\Psi_q\,(x1, x2) = u\,(x1, x2) + i v_1\,(x1, x2) + j v_2\,(x1, x2) + k v\,(x1, x2)$ where v_1 , v_2 and v are the partial and total Hilbert Transforms .

The real part of this quaternionic signal is also the original signal $u(x1, x2)$

The polar representation define a single local amplitude A_q and three phase functions Φ_i ,Φ_j and Φ_k

$$\Psi_q\ (x1, x2) = A_q\, e^{\Phi i}\, e^{\Phi j}\, e^{\Phi k}$$

The amplitude is Aq $(x_1,x_2)= \sqrt{u^2+v^2+v_1^2+v_2^2}$

The phase angles are :

Φ_i, Φ_J, Φ_k

II- 3 Monogenic 2D signal and Riesz transform

The notion of the monogenic 2D signal is derived using Riesz transform ;The monogenic signal Φ_m (x_1,x_2) can be defined by the inverse QFT of the four quadrant quaternionic-valued spectrum (**U_q**) of the original signal $u(x_1,x_2)$.

The monogenic 2D signal Φ_m (Ref 3) has the form:

$$\Phi_m\,(x_1,x_2) = u(x_1,x_2) + i v_{r1} + j\, v_{r2}$$

The Riesz transform v_{r1} and v_{r2} of the real signal $u(x_1,x_2)$ have the form of convolution of u with the Riesz kernel r_1 and r_2.

$$v_{r1}(x_1,x_2) = u(x_1,x_2) ** r_1(x_1,x_2)$$
$$v_{r2}(x_1,x_2) = u(x_1,x_2) ** r_2(x_1,x_2)$$

$r_1(x_1,x_2)= \dfrac{x_1}{2\pi\left[\sqrt{x_1^2+x_2^2}\right]^3}$ in space domain , equivalently to $\dfrac{f_1}{\left[\sqrt{f_1^2+f_2^2}\right]}$,cosine in Fourier domain.

$r_2(x_1,x_2)=\dfrac{x_2}{2\pi\left[\sqrt{x_1^2+x_2^2}\right]^3}$ in space domain, equivalently to $\dfrac{f_2}{\left[\sqrt{f_1^2+f_2^2}\right]}$,sine in Fourier domain

QFT $(i\nu_{r1})$= $\mathbf{U_q}\ \dfrac{f_1}{\left[\sqrt{f_1^2+f_2^2}\right]}$

QFT $(j\,\nu_{r2})$= $\mathbf{U_q}\ \dfrac{f_2}{\left[\sqrt{f_1^2+f_2^2}\right]}$

The quaternionic spectrum of the monogenic signal $\Phi_m(x_1,x_2)$ is :

$$\text{QFT}\left[\Phi_m(x_1,x_2)\right] = \mathbf{U_q}\left[1+\frac{f_1+f_2}{\left[\sqrt{f_1^2+f_2^2}\right]}\right]$$

The parameters of the monogenic 2D signal Φ_m are :

$A_m=\sqrt{u^2+vr_2^{\,2}+vr_1^{\,2}}$, Amplitude

$\theta=\text{Arc tg}\ \sqrt{vr_2^{\,2}+vr_1^{\,2}}\ \times\ \dfrac{1}{u}$ local phase

$\phi=\text{Arc tg}\ \dfrac{vr_2}{vr_1}$ orientation angle

$$\Phi_m(x_1,x_2)=u(x_1,x_2)+iv_{r1}(x_1,x_2)+jv_{r2}(x_1,x_2).$$

$$\Phi_m(x_1,x_2)= A_m\ (\cos\theta+i\sin\theta\cos\phi+j\sin\theta\sin\phi)$$

The spherical coordinate representation of the monogenic signal $\Phi_m(x_1,x_2)$ define a single local amplitude A_m and two local angle called the phase function θ and the orientation angle ϕ.

Remark: The amplitude is isotropic, and no depends of the orientation angle **ϕ**

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Marcel Riesz (1886-1969), Hungarian mathematician, has worked in Sweden , specialist of number theory and Clifford algebra. He is the younger brother of Frederic Riesz , also great recognized mathematician.

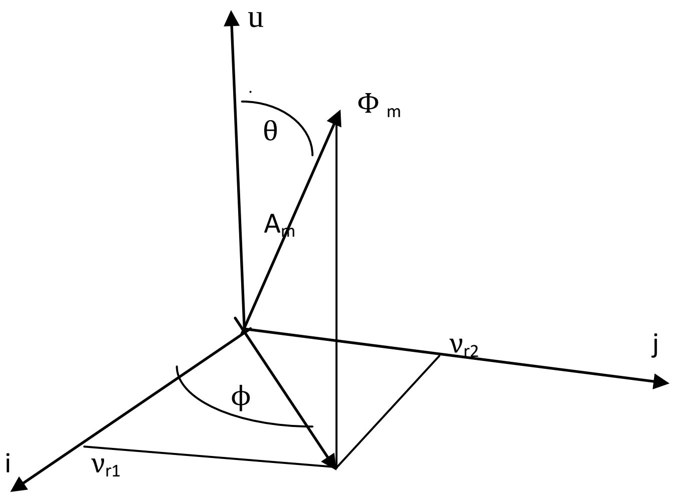

Fig 3 : 3D representation of the three parameters of Φ_m : A_m ,θ , ϕ

III –Generalization : 3D Hahn single-orthant A_s,multiquaternions (Ref 4).

Hermitian symmetry is a property of the 3D Fourier transform. We can use this property to compute the 3D analog signal (As). As shown in FIG. 4, the frequency space 3D has eight (2^3) orthants. On the basis of the Hermitian symmetry of the Fourier transformation, the input signal 3D can be recovered from any half-plane spectrum. Without loss of generality, it is possible to work on the half-space spectra with f1> 0 to calculate the analytic signal with a single orthant, that is to say the orthants with the labels I, III, V and VII (Fig 4).The 3D real signal can be represented by four ($2^3/2$) analytic signals.

For each analytic signal simple orthant, one can calculate its modulus and the phase of its polar form. A total of 4x2 parameters are obtained for each voxel of the original 3D image.

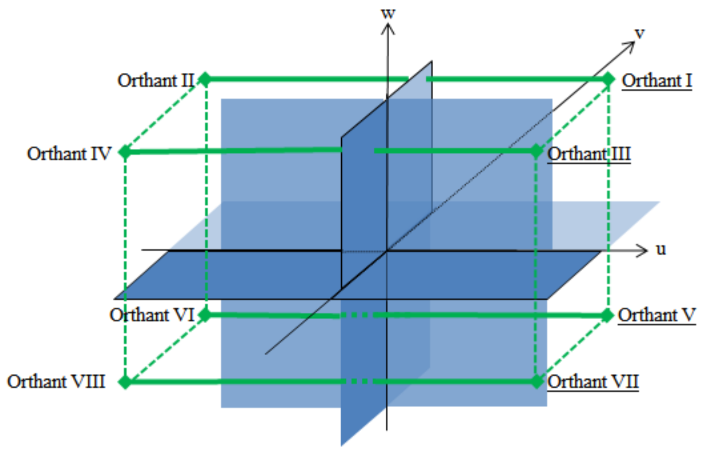

Fig 4 Frequency spectrum for 3D images (Ref L Wang Thesis Lyon (2014))

IV -Interest of the different representations

As has been defined in 1992 (Ref 5) and the Qs and Ms about 10 years later (Ref 3, 7)

With As, the polar representation defines two local amplitudes and two local phase functions The real signal can be recovered using the formula :

$$u(x_1,x_2)=1/2\ [A_1\cos\phi_1 + A_2\cos\phi_2]$$

With Qs , the polar representation defines a single local amplitude and three phases functions .The reconstruction formula for the Qs is:

$$u(x_1,x_2)=A_Q[\cos\Phi_i \cos\Phi_j\cos\Phi_k - \sin\Phi_i\sin\Phi_j\sin\Phi_k]$$

Therefore, the total number of functions equals four, as in the case of As

Moreover, there are closed expressions enabling the calculation of the quaternionic functions starting with analytic functions

In consequence, the As and Qs are equivalent methods .

Differently to the single quadrant spectra of As and Qs the QFT of the Ms has a four – quadrant support. The spherical coordinate representation of the Ms defines a single local amplitude and two local angles called the phase function $u(x_1,x_2)=$ and the orientation angle ϕ.The real signal can be recovered using the formula:

$$u(x_1,x_2)=A_m \cos\theta \quad \text{witch is independent on } \phi$$

V-Conclusions

For practical applications, such as the analysis of the structural properties of 2D tomographic images (Ref 8) , the Monogenic approach Ms is, for its simplicity of interpretation and its very condensed form, the most effective tool and the easiest to implement

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