RING MORPHISMS ON STRONGLY MAGIC SQUARES

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ABSTRACT

Several aspects of magic(al) square studies fall within the computational universe. Experimental computation has revealed patterns, some of which have lead to analytic insights, theorems or combinatorial results. Other numerical experiments have provided statistical results for some very difficult problems. Magic squares generally fall into the realm of recreational mathematics [4, 5], however a few times in the past century and more recently, they have become the interest of more-serious mathematicians. The unique normal square of order three was known to the ancient Chinese, who called it the Lo Shu. A version of the order-4 magic square with the numbers 15 and 14 in adjacent middle columns in the bottom row is called Dürer's magic square. A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number, known as the magic constant. The paper discuss about a well-known class of magic squares; the strongly magic square. The strongly magic square is a magic square with a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant. In this paper a generic definition for Strongly Magic Squares is given. A function on strongly magic squares is also defined and it is proved to be a group homomorphism and isomorphism. The paper also sheds light on the ring isomorphism of strongly magic squares

Keywords: Magic Square, Magic Constant, Strongly Magic Square, Homomorphism, Isomorphism, Ring morphism.

INTRODUCTION

Magic squares have turned up throughout history, some in a mathematical context and others in religious contexts. Magic squares date back to the first millennium B.C.E in China [1]. It was a magic square of order three thought to have appeared on the back of a turtle emerging from a river. Other magic squares surfaced at various places around the world in the centuries following their discovery. Some of the more interesting examples were recorded in Europe during the 1500s. Cornelius Agrippa wrote De Occulta Philosophia in 1510 [2] Magic squares generally fall into the realm of recreational mathematics [3, 4], however a few times in the past century and more recently, they have become the interest of more-serious mathematicians.

A normal magic square is a square array of consecutive numbers from $1 \ldots n^2$ where the rows, columns, diagonals and co-diagonals add up to the same number. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square of order 4 have a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant [5]. There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possess advanced mathematical properties.

NOTATIONS AND MATHEMATICAL PRELIMINARIES

Magic Square

A magic square of order n over a field R where R denotes the set of all real numbers is an n[th] order matrix $[a_{ij}]$ with entries in R such that

$$\sum_{j=1}^{n} a_{ij} = \rho \qquad for \ \ i = 1,2, \ldots.. n \qquad (1)$$

$$\sum_{j=1}^{n} a_{ji} = \rho \qquad for \ \ i = 1,2, \ldots.. n \qquad (2)$$

$$\sum_{i=1}^{n} a_{ii} = \rho \ , \ \sum_{i=1}^{n} a_{i,n-i+1} = \rho \qquad (3)$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant. [8]

Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$. In the example given below the magic constant of A is 15 and B is 34

A =

8	1	6
3	5	7
4	9	2

B =

9	16	5	4
7	2	11	14
12	13	8	1
6	3	10	15

Strongly magic square (SMS): Generic Definition

A strongly magic square over a field R is a matrix $[a_{ij}]$ of order $n^2 \times n^2$ with entries in R such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \ \ for \ \ i = 1,2, \ldots.. n^2 \qquad (4)$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \ \ for \ \ i = 1,2, \ldots.. n^2 \qquad (5)$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho \ , \qquad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \qquad (6)$$

$$\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \;\; for \;\; i,j = 1,2,\ldots..n^2 \qquad (7)$$

$$where\ the\ subscripts\ are\ congruent\ modulo\ n^2$$

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal & co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as ${M_{0C}}^{(n)} or {M_{0R}}^{(n)}$ and ρ is the magic constant.

Note: The nth order subsquare sum with k column gaps or k row gaps is generally denoted as ${M_{kC}}^{(n)}$ or ${M_{kR}}^{(n)}$ respectively.

Group homomorphism

A mapping ϕ from a group $< G,* >$into a group $< G', *' >$ is a homomorphism of G into G$'$ if

$\phi(a * b) = \phi(a) *' \phi(b)$ for all $a, b \in$ G [9]

Group isomorphism

A one to one onto homomorphism ϕ from a group $< G,* >$into a group $< G', *' >$ is defined as isomorphism [9]

A one to one and onto mapping

A function $\phi : X \to Y$ is one to one if $\phi(x_1) = \phi(x_2)$ only when $x_1 = x_2$.
The function ϕ is onto of Y if the range of ϕ is Y

Other Notations

1. R denotes the set of all real numbers.
2. S denote the set of all strongly magic squares of order $n^2 \times n^2$
3. S_a denote the set of all strongly magic squares of order $n^2 \times n^2$ denote the set of all strongly magic squares of the form $[a_{ij}]_{n^2\times n^2}$ such that $a_{ij} = a$ for every $i,j = 1,2,\ldots n^2$. Here A is denoted as $[a]$, i.e. If $A \in S_a$ then $\rho(A) = n^2 a$

PROPOSITIONS AND THEOREMS

Proposition 3.1

If A and B are two Strongly magic squares of order $n^2 \times n^2$ with $\rho(A) = a$ and $\rho(B) = b$, then $C = (\lambda + \mu)(A + B)$ is also a Strongly magic square with magic constant $(\lambda + \mu)\big(\rho(A) + \rho(B)\big)$; for every $\lambda, \mu \in R$

Proof:

Let $A = [a_{ij}]_{n^2\times n^2}$ and $B = [b_{ij}]_{n^2\times n^2}$

Then $C = (\lambda + \mu)(A + B)$

$\qquad = [(\lambda + \mu)(a_{ij} + b_{ij})]$

Sum of the ith row elements of

$$C = \sum_{j=1}^{n^2} c_{ij} = (\lambda + \mu)\left(\sum_{j=1}^{n^2}(a_{ij}) + \sum_{j=1}^{n^2}(b_{ij})\right)$$
$$= (\lambda + \mu)(a + b)$$
$$= (\lambda + \mu)(\rho(A) + \rho(B))$$

A similar computation holds for column sum, diagonals sum and sum of the $n \times n$ sub squares

From the above propositions the following results can be obtained by putting suitable values for λ, and μ

RESULTS

If for every $\lambda, \mu \epsilon R$ *and* $A, B \in S$,

1.1) $\lambda(A + B) \in S$ *with* $\rho(\lambda(A + B)) = \lambda(\rho(A) + \rho(B))$

1.2) $(A + B) \in S$ *with* $\rho(A + B) = \rho(A) + \rho(B)$

1.3) $\lambda A \in S$ with $\rho(\lambda A) = \lambda\ \rho(A)$

1.4) $\lambda A + \mu B \in S$ with $\rho(\lambda A + \mu B) = \lambda\ \rho(A) + \mu\ \rho(B)$

1.5) – A ∈ S with – A ∈ S

Theorem 3.2

$< S, + >$ forms an abelian group.

Proof:

I. Closure property: *if* $A, B \in S$, *then* $A + B \in S$. (*from above result* 1.2)

II. Associativity : *if* $A, B, C \in S$, *then* $A + (B + C) = (A + B) + C \in S$ (Since matrix addition is associative.)

III. Existence of Identity: There exists 0 matrix in S *so that* $A + 0 = 0 + A = A$, where 0 acts as the identity element.

IV. Existence of additive inverse : For every $A \in S$, there exists $-A \in S$ so that $A + (-A) = 0$ where $0 \in S$ (*from result* 1.5).

V. Commutativity : If $A, B \in S$, then $A + B = B + A \in S$ (Since matrix addition is commutative .)

This completes the proof.

Proposition 3.3

S_a forms a subgroup of the abelian group S.

Proof:

It is clear that $S_a \subset S$.

For $A, B \in S_a$; $A = [a]$ and $B = [b]$, then clearly $A - B = [a - b] \in S_a$

Thus S_a forms a subgroup of the abelian group S.

Proposition 3.4

The mapping $\phi : S_a \to R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S_a$ is a group homomorphism

Proof

Let $A, B \in S_a$; then $\phi(A + B) = \rho(A + B) = \rho(A) + \rho(B)$

(By Result 1.2 and Proposition 3.3)

$$= \phi(A) + \phi(B)$$

Proposition 3.5

The mapping $\phi : S_a \to R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S_a$ is a group isomorphism.

Proof

Let $,B \in S_a$; $A = [a]$, $B = [b]$ then $\rho(A) = n^2a$ and $\rho(B) = n^2b$

To show that ϕ is 1-1

$$\phi(A) = \phi(B)$$
$$\Rightarrow \rho(A) = \rho(B)$$
$$\Rightarrow n^2a = n^2b$$
$$\Rightarrow a = b$$

To show that ϕ is onto

For every $a \in R$, there exists $A = \left[\frac{a}{n^2}\right] \in S_a$ such that $\rho(A) = a$.

Since ϕ is 1-1 and onto and from Proposition 3.4,result can be deduced.

Proposition 3.6

The mapping $\phi : S_a \rightarrow R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S_a$ is a ring homomorphism.

Proof:

Let $A , B \in S_a$,then

$$\phi(A + B) = \rho(A + B)$$
$$= \rho(A) + \rho(B) \text{ (By Results 1.2)}$$
$$= \phi(A) + \phi(B)$$

Now $AB = [n^2a\,b]$ $with\ \rho\ (AB) = n^4a\,b$

$$\phi(AB) = \rho(AB)$$
$$\rho(AB) = n^4a\,b$$
$$= n^2a\, n^2b$$
$$= \rho(A)\rho(B)$$
$$= \phi(A)\phi(B)$$

Proposition 3.7

The mapping $\phi : S_a \rightarrow R$ defined by $\phi(A) = \rho(A)$, $\forall A \in S$ is a ring isomorphism.

Proof:

From Proposition 3.6 and Proposition 3.5 it can be deduced.

CONCLUSION

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different mathematical courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described. Physical application of magic squares is still a new topic that needs to be explored more. There are many interesting ideas for research in this field.

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