A COMPARISON OF MEMBRANE SHELL THEORIES OF HYBRID ANISOTROPIC MATERIALS

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ABSTRACT

Membrane shell theories are simple and widely used but also care must be taken to prevent secondary bending moments due to the unbalanced arrangement of laminates of anisotropic materials. At times, bending theory may have to be adopted and the current design codes, such as ASME, API and ACI must be reviewed for the case of anisotropic materials. The stresses and strains can be significantly different between the pure membrane and bending theories. This paper derives a membrane type shell theory of hybrid anisotropic materials, governing differential equations together with the procedures to locate the mechanical neutral axis. The theory is derived by first considering generalized stress strain relationship of a three dimensional anisotropic body which is subjected to 21compliance matrix and then non-dimensionalizing each variable with asymptotically expansion. After applying to the equilibrium and stress-displacement equations, we are allowed to proceed asymptotic integration to reach the first approximation theory. Also possible secondary moments due to the unbalanced built up of lamination are quantifiably expressed. The theory is different from the so called pure membrane or the semi-membrane analysis.

Key Words: Hybrid anisotropic materials; Asymptotic integration; Length scales; Membrane Stresses; Secondary Bending moments.

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INTRODUCTION

Shell theories and its design and manufacturing technology are becoming more important recently as the outer space exploration being more active. It ranges from deep water submarines, space vehicles, to the dome type human residences in the Moon or Mars. The membrane theory of shell is simple and been existing for generations since Trusdell and Goldenveiser have theoretically formulated as shown in the References (7) and (8).

The mechanics of composites are complicated compared to the ordinary conventional materials such as steel and other metallic brands but composites possess such characteristics as high strength/density and modulus/density ratios, which will allow flight vehicles more efficient and increased distance. The filaments embedded in the matrix materials of composites give additional stiffness and tensile strength. They can be arranged arbitrarily so as to make a structure more resistant to loadings. As the mechanical properties of composites vary depending on the direction of the fiber arrangement, it is necessary to analyze them by use of an anisotropic theory. Also the current design codes including ASME, API and ACI, References (15) to (18), are all based on membrane theory for isotropic materials.

Pressure vessels of composite materials are, in general, constructed of thin layers of different thickness with different material properties. The properties of anisotropic materials are represented by different elastic coefficients and different cross-ply angles. The cross-ply angle, γ, is the angle between major elastic axis of the material and reference axis (Figure 1 and 2). The variation in properties in the direction of the thickness implies non-homogeneity of the material and composite structures must thus be analyzed according to theories which allow for non-homogeneous anisotropic material behavior. Our task is to formulate a theory for a shell of composite materials which are non-homogeneous and anisotropic materials.

According to the exact three-dimensional theory of elasticity, a shell element is considered as a volume element. All possible stresses and strains are assumed to exist and no simplifying assumptions are allowed in the formulation of the theory. We therefore allow for six stress components, six strain components and three displacements as indicated in the following equation:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad i, j = 1, 2, 3 \qquad k, l = 1, 2 \tag{1}$$

where σ_{ij} and ε_{kl} are stress and strain tensors respectively and C_{ijkl} are elastic moduli. There are thus a total of fifteen unknowns to solve for in a three dimensional elasticity problem. On the other hand, three equilibrium equations and six strain displacement equations can be obtained for a volume element and six generalized Hook's law equations can be used. A total of fifteen equations can thus be formulated and it is basically possible to set up a solution for a three-dimensional elasticity problem. It is however very complicated to obtain a unique solution which satisfies both the above fifteen equations and the associated boundary conditions. This led to the development of various theories for structures of engineering interest. A detailed description of classical shell theory can be found in various references [1-12].

In the first part of this article, the asymptotic expansion and integration method is used to reduce the exact three-dimensional elasticity theory for a non-homogeneous, anisotropic cylindrical shell to approximate theories. The analysis is made such that it is valid for materials which are non-homogeneous to the extent that their mechanical properties are allowed to vary with the thickness coordinate. The derivation of the theories is accomplished by first introducing the shell dimensions and as yet unspecified characteristic length scales via changes in the independent variables. Next, the dimensionless stresses and displacements are expanded asymptotically by using the thinness of the shell as the expansion parameter. A choice of characteristic length scales is then made and corresponding to different combination of these length scales, different sequences of systems of differential equations are obtained. Subsequent integration over the thickness and satisfaction of the boundary conditions yields the desired equations governing the formulation of the first approximation stress states of non-homogeneous anisotropic cylindrical shell.

Formulation of Cylindrical Shell theory of Anisotropic Materials

Consider a non-homogeneous, anisotropic volume element of a cylindrical body with longitudinal, circumferential (angular) and radial coordinates being noted as z, θ, r, respectively and subjected to all possible stresses and strains (Figure 1). The cylinder occupies the space between

$a \le r \le a+h$ and the edges are located at $z=0$ and $z=L$. Here, a is the inner radius, h the thickness and L the length.

Assuming that the deformations are sufficiently small so that linear elasticity theory is valid, the following equations govern the problem:

$$\begin{aligned}
&\left(r\tau_{rz}\right)_{,r}+\tau_{\theta z,\theta}+\left(r\sigma_z\right)_{,z}=0\\
&\left(r\tau_{r\theta}\right)_{,r}+\sigma_{\theta,\theta}+\left(r\tau_{\theta z}\right)_{,z}+\tau_{\theta z}=0\\
&\left(r\sigma_r\right)_{,r}+\tau_{r\theta,\theta}+\left(r\tau_{rz}\right)_{,z}-\sigma_\theta=0
\end{aligned} \tag{2}$$

$$\begin{aligned}
&u_{z,z}=S_{11}\sigma_z+S_{12}\sigma_\theta+S_{13}\sigma_r+S_{14}\tau_{r\theta}+S_{15}\tau_{rz}+S_{16}\tau_{\theta z}\\
&\frac{1}{r}\left(u_{\theta,\theta}+u_r\right)=S_{12}\sigma_z+S_{22}\sigma_\theta+\cdots+S_{26}\tau_{\theta z}\\
&u_{r,z}=S_{13}\sigma_z+\cdots+S_{36}\tau_{\theta z}\\
&\frac{1}{r}u_{r,\theta}+u_{\theta z}-\frac{1}{r}u_\theta=S_{14}\sigma_z+\cdots+S_{16}\tau_{\theta z}\\
&u_{z,r}+u_{r,z}=S_{15}\sigma_z+\cdots+S_{36}\tau_{\theta z}\\
&u_{\theta,z}+\frac{1}{r}u_{z,\theta}=S_{16}\sigma_z+\cdots+S_{66}\tau_{\theta z}
\end{aligned} \tag{3}$$

In the above Equations (2) are equilibrium equations and (3) stress-displacement relations. In that u_r, u_θ, u_z are the displacement components in the radial, circumferential and longitudinal directions, respectively, σ_r, σ_θ, σ_z the normal stress components in the same directions and $\tau_{\theta z}$, τ_{rz}, $\tau_{r\theta}$ are the shear stresses on the θ-z face, r-z face, r-θ face respectively (Figure 1). A comma indicates partial differentiation with respect to the indicated coordinates. The S_{ij} ' s $\left(i,j=1,2,\ldots\ldots,6\right)$ in the equation (3) are the components of compliance matrix and represent the directional properties of the material. Complete anisotropy of the material is allowed for and there are thus 21 independent material constants. We are not allowed to illuminate any of those components since the material properties are depending on the manufactures set up and different gravity environment in case of aerospace vehicles. Also the compliance matrix is symmetric, $S_{ij}=S_{ji}$, and the components can be expressed in terms of engineering constants as follows:

$$\begin{aligned}
&S_{ii}=\frac{1}{E_i},\ \left(i=1,3\right)\\
&S_{ij}=\frac{-\nu_{ij}}{E_i},\ \left(i=1,2,\ j=2,3,\ i\neq j\right)\\
&S_{44}=\frac{1}{G_{23}}
\end{aligned} \tag{4}$$

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$$S_{55} = \frac{1}{G_{13}}$$

$$S_{66} = \frac{1}{G_{12}}$$

In equation (4) the E_i ' s are the Young's moduli in tension along the i – direction and ν_{ij} and G_{ij} are the Poisson's ratio and shear moduli in the i-j face, respectively. Equation (4) implies anisotropic property of the material only, material to be non-homogeneous, different properties of each layer of the shell, we will allow the material property variation in the radial direction as follows:

$$S_{ij} = S_{ij}(r) \tag{5}$$

The above equation is unique and different from most of conventional theories, including Reddy's, Reference (11), which input the engineering constants artificially from the beginning, while we take the existence and magnitude of components only by approximation theory of the asymptotic expansion.

The principal material axes (r', θ', z') in general do not coincide with the body axes of the cylindrical shell (r, θ, z). If the material properties S'_{ij} with respect to material axes specified, then the properties with respect to the body axes are given by the following transformation equations:

$$\begin{aligned}
S_{11} &= S'_{11}\cos^4\gamma + \left(2S'_{12} + S'_{66}\right)\sin^2\gamma\cos^2\gamma + S'_{22}\sin^4\gamma \\
&\quad + \left(S'_{16}\cos^2\gamma + S'_{26}\sin^2\gamma\right)\sin 2\gamma , \\
S_{22} &= S'_{11}\cos^4\gamma + \left(2S'_{12} + S'_{66}\right)\sin^2\gamma\cos^2\gamma + S'_{22}\cos^4\gamma \\
&\quad - \left(S'_{16}\sin^2\gamma + S'_{26}\cos^2\gamma\right)\sin 2\gamma , \\
S_{12} &= S'_{12} + \left(S'_{11} + S'_{22} - 2S'_{12} - S'_{66}\right)\sin^2\gamma\cos^2\gamma \\
&\quad + \tfrac{1}{2}\left(S'_{26} - S'_{16}\right)\sin 2\gamma\cos 2\gamma , \\
S_{66} &= S'_{66} + 4\left(S'_{11} + S'_{22} - 2S'_{12} - S'_{66}\right)\sin^2\gamma\cos^2\gamma + 2\left(S'_{26} + S'_{16}\right)\sin 2\gamma\cos 2\gamma , \\
S_{16} &= \left[S'_{22}\sin^2\gamma - S'_{11}\cos^2\gamma + \tfrac{1}{2}\left(2S'_{12} + S'_{66}\right)\cos 2\gamma\right]\sin 2\gamma + S'_{16}\cos^2\gamma\left(\cos^2\gamma - 3\sin^2\gamma\right) \\
&\quad + S'_{26}\sin^2\gamma\left(3\cos^2\gamma - \sin^2\gamma\right), \\
S_{26} &= \left[S'_{22}\cos^2\gamma - S'_{11}\sin^2\gamma + \tfrac{1}{2}\left(2S'_{12} + S'_{66}\right)\cos 2\gamma\right]\sin 2\gamma \\
&\quad + S'_{16}\sin^2\gamma\left(3\cos^2\gamma - \sin^2\gamma\right) + S'_{26}\cos^2\gamma\left(\cos^2\gamma - 3\sin^2\gamma\right).
\end{aligned} \tag{6}$$

where γ *is* the angle of anisotropic orientation between the z' and the original coordinate z axes. For the case of an orthotropic material, where the major and minor elastic axes are 90 Degree, the transformation equations (6) are reduced to equation (7):

$$
\begin{aligned}
&\frac{1}{E_2^{'}}=\frac{\cos^4\gamma}{E_1}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\sin^2\gamma\cos^2\gamma+\frac{\sin^4\gamma}{E_2},\\
&\frac{1}{E_2^{'}}=\frac{\sin^4\gamma}{E_1}+\left(\frac{1}{G}-\frac{2\nu_1}{E_1}\right)\sin^2\gamma\cos^2\gamma+\frac{\cos^4\gamma}{E_2},\\
&\frac{1}{G^{'}}=\frac{1}{G}+\left(\frac{1+\nu_1}{E_2}+\frac{1-\nu_1}{E_2}-\frac{1}{G}\right)\sin^2 2\gamma\\
&\nu_1^{'}=E_1^{'}\left[\frac{\nu_1}{E_1}-\frac{1}{4}\left(\frac{1+\nu_1}{E_1}+\frac{1+\nu_1}{E_2}-\frac{1}{G}\right)\right]\\
&\nu_2^{'}=\nu_1^{'}\frac{E_2^{'}}{E_1^{'}}
\end{aligned}
\qquad (7)
$$

The invariants are expressed by

$$\left(1/E_1^{'}\right)+\left(1/E_2^{'}\right)-\left(2\nu_1^{'}/E_1^{'}\right)=\left(1/E_1\right)+\left(1/E_2\right)-\left(2\nu_1^{'}/E_1\right)$$

$$\left(1/G^{'}\right)+\left(4\nu_1^{'}/E_1^{'}\right)=\left(1/G\right)+\left(4\nu_1/E_1\right)$$

The shell is subjected to a uniformly distributed tensile force then the boundary conditions are as follows:

$$
\begin{aligned}
&\sigma_r=\tau_{r\theta}=\tau_{rz}=0 && (r=a)\\
&\sigma_r=\rho(\theta,z),\tau_{r\theta}=\tau_{rz}=0 && (r=a+h)
\end{aligned}
\qquad (8)
$$

We will find it convenient to work with stress resultants rather than the stresses themselves. These stress resultants which are forces and moments per unit length are obtained by integrating with respect to the thickness coordinate. They are:

$$
\begin{aligned}
&N_z=\int_a^{a+h}\sigma_z\left[1+\frac{r-a-d}{a+d}\right]dr\\
&N_\theta=\int_a^{a+h}\sigma_\theta dr\\
&N_{\theta z}=\int_a^{a+h}\tau_{\theta z}dr\\
&N_{z\theta}=\int_a^{a+h}\tau_{\theta z}\left[1+\frac{r-a-d}{a+d}\right]dr\\
&M_\theta=\int_a^{a+h}a_\theta\left[r-a-d\right]dr
\end{aligned}
\qquad (9)
$$

$$M_z = \int_a^{a+h} a_z \left[\frac{r-a-d}{a+d} \right] rdr$$
$$M_{\theta z} = \int_a^{a+h} \tau_{\theta z} \left[r-a-d \right] dr$$
$$M_{\theta z} = \int_a^{a+h} \tau_{\theta z} \left[\frac{r-a-d}{a+d} \right] rdr$$

In the equations (9) variable *a* denotes the inner radius of the cylindrical shell and *d* the distance from the inner surface to the reference surface where the stress resultants are defined. Note that $N_{\theta z}$ and $N_{z\theta}$ and $M_{z\theta}$ and $M_{\theta z}$ respectively are different. This is due to the fact that the terms of the order of thickness over radius are not neglected compare to one in the integral expressions.

Formulation of a Boundary Layer Theory

The procedure used to formulate the shell theory here is basically to reduce the three dimensional equations to two dimensional thin shell equations and we will use the asymptotic integration of the equations (2) and (3) describing the cylindrical shell. As a first step to integrating equations (2) and (3), we make them non-dimensionalized coordinates as follows:

$$X = z/L\,, \qquad Y = (r-a)/h\,, \qquad \varphi = \theta/\beta \tag{10}$$

where L and $\ell(=\beta a)$ are quantities which are to be determined later.

Next the compliance matrix, the stresses and deformations are non-dimensionalized by the use of a representative stress level σ, a representative material property S and the shell radius a, as follows:

$$\begin{aligned}
&S_{ij} = S\overline{S}_{ij} \\
&\sigma_z = \sigma\tau_{r\theta}\,, \qquad \sigma_\theta = \sigma\tau_\theta\,, \qquad \sigma_r = \sigma\tau_r \\
&\tau_{r\theta} = \sigma\tau_{r\theta}\,, \qquad \tau_{rz} = \sigma\tau_{rz}\,, \qquad \tau_{\theta z} = \sigma\tau_{\theta z} \\
&u_r = \sigma aSv_r\,, \quad u_\theta = \sigma aSv_\theta\,, \quad u_z = \sigma aSv_z
\end{aligned} \tag{11}$$

where the dimensionless displacements and stresses are functions of x, y and φ. These variables together with their derivatives with respect to x, y and φ are assumed to be of order unity. The parameters L and ℓ introduced in equation (10) are thus seen to be characteristic length scales for changes of the stresses and displacements in the axial and circumferential directions, respectively.

It is convenient at this point to define what is here meant by the concept of relative order of magnitude. Consider a small parameter ε, ε is less than 1. With respect to an arbitrary domain D of the cylinder, M_1 is said to be of order ε^n relative to a second quantity M_2

$$M_2 \approx \varepsilon^n M_1 \tag{12}$$

if everywhere in D (with the possible exception of some isolated small regions) the relationship

$$\varepsilon^{n+m} \leq |M_2|/|M_1| \leq \varepsilon^{n-m} \tag{13}$$

holds for a suitably chosen value of m, $0<m<1$. According to this definition, two quantities are of the same order if $n=0$ in the above, while a quantity is of order unity when $n=0$ and $M_1=1$. Substitution of the dimensionless variables defined by (10) and (11) into the elasticity equations (2) and (3) yields the following dimensionless equations:

Stress-displacement relationships are

$$\begin{aligned}
&v_{r,y} = \lambda\left[\bar{S}_{31}t_z + \bar{S}_{32}t_\theta + \bar{S}_{33}t_r + \bar{S}_{34}t_{r\theta} + \bar{S}_{35}t_{rz} + \bar{S}_{36}t_{\theta z}\right] \\
&v_{z,y} + \lambda\left(\frac{a}{L}\right)v_{r,x} = \lambda\left[\bar{S}_{51}t_z + \bar{S}_{52}t_\theta + \bar{S}_{53}t_r + \bar{S}_{54}t_{r\theta} + \bar{S}_{55}t_{rz} + \bar{S}_{56}t_{\theta z}\right] \\
&\lambda v_{r,\varphi} + \left(\frac{l}{a}\right)(1+\lambda y)v_{\theta,y} - \left(\frac{l}{a}\right)\lambda v_\theta \\
&= \lambda\left(\frac{l}{a}\right)(1+\lambda y)\left[\bar{S}_{41}t_z + \bar{S}_{42}t_\theta + \bar{S}_{43}t_r + \bar{S}_{44}t_{r\theta} + \bar{S}_{45}t_{rz} + \bar{S}_{46}t_{\theta z}\right] \\
&v_{z,r} = \left(\frac{L}{a}\right)\left[\bar{S}_{11}t_z + \bar{S}_{12}t_\theta + \bar{S}_{13}t_r + \bar{S}_{14}t_{r\theta} + \bar{S}_{15}t_{rz} + \bar{S}_{16}t_{\theta z}\right] \\
&\left(\frac{a}{l}\right)v_{\theta,\varphi} + v_r = (l+\lambda y)\left[\bar{S}_{21}t_z + \bar{S}_{22}t_\theta + \bar{S}_{23}t_r + \bar{S}_{24}t_{r\theta} + \bar{S}_{25}t_{rz} + \bar{S}_{26}t_{\theta z}\right] \\
&\left(\frac{a}{L}\right)(l+\lambda y)v_{\theta,z} + \left(\frac{a}{l}\right)v_{z,\varphi} = (l+\lambda y)\left[\bar{S}_{61}t_z + \bar{S}_{62}t_\theta + \bar{S}_{63}t_r + \bar{S}_{64}t_{r\theta} + \bar{S}_{65}t_{rz} + \bar{S}_{66}t_{\theta z}\right]
\end{aligned} \tag{14}$$

Equilibrium equations are expressed as

$$\begin{aligned}
&\left[t_{rz}(1+\lambda y)\right]_{,y} + \left(\frac{\lambda a}{l}\right)t_{\theta z},\phi + \left(\frac{\lambda a}{L}\right)(1+\lambda y)t_{z,x} = 0 \\
&\left[t_{r\theta}(1+\lambda y)\right]_{,y} + \left(\frac{\lambda a}{l}\right)t_{\theta,\phi} + \lambda t_{r\theta} + \left(\frac{\lambda a}{L}\right)(1+\lambda y)t_{\theta z,x} = 0 \\
&\left[t_r(1+\lambda y)\right]_{,y} + \left(\frac{\lambda a}{l}\right)t_{r\theta,\phi} + \left(\frac{\lambda a}{L}\right)(1+\lambda y)t_{rz,x} - \lambda t_\theta = 0
\end{aligned} \tag{15}$$

where λ is the thin shell parameter defines as

$$\lambda = h/a \tag{16}$$

The parameter λ is representative of the thinness of the cylindrical shell.

λ is much less than 1 (17)

The dimensionless coefficients S_{ij} of the compliance matrix in general are not all of same order.
We therefore assume that they can be expanded in terms of finite sum as:

$$\overline{S_{ij}}\left(y_i\lambda\right) = \sum_{n=0}^{N} S_{ij}^{(n)}\left(y\right)\lambda^{\frac{n}{2}} \tag{18}$$

where the $S_{ij}^{(n)}\left(y\right)$ are of order unity or vanish identically. Next, we assume that each displacement components, represented by the generic symbol $\nu^{(m)}$, and each stress components represented by the generic symbol $\tau^{(m)}$, can be expanded in terms of a power series in $\lambda^{1/2}$

$$\nu\left(y,x,\phi;\lambda\right) = \sum_{m=0}^{M} \nu^{(m)}\left(y,x,\phi\right)\lambda^{m/2} \tag{19}$$

$$t\left(y,x,\phi;\lambda\right) = \sum_{m=0}^{M} t^{(m)}\left(y,x,\phi\right)\lambda^{m/2} \tag{20}$$

The $\nu^{(m)}$ and $t^{(m)}$ are of order unity. No convergence properties are assumed for series (19) only asymptotic validity for λ. That is, if expansions (19) are terminated at some power of $\lambda^{1/2}$, the error in using the expansions rather than the exact solutions tends to zero as λ approaches zero. Length scale L and ℓ are as yet arbitrary. Their choice, as will be seen in the subjects to follow, determines the type of shell theory to be identified.

Last step in the procedure consists of substituting expansions of the series and one of assumed length scales into the dimensionless elasticity equations of stress-displacement and equilibrium given by equations (14) and (15). Upon selecting terms of like powers in $\lambda^{1/2}$ on both sides of each equations and requiring that the resulting equations be integrable with respect to the thickness coordinate and be capable be capable of identifying the relations for all stresses and displacement components, we will obtain systems of differential equations. The first system of equations of "thin shell" theory and we will call it the first approximation system. We can however obtain stresses and displacements of each layer of thickness coordinate, that can be an advantage of the procedure among others. In the following section, the thin shell theories for different combinations of length scales can be derived.

Formulation of Membrane Type Theory

(Associated with characteristic length scales, a)

As we observed the shell geometry is an important factor for the formulation of theories. The basic geometry of cylindrical shell are the longitudinal length L, inside radius a, total wall thickness h and the distance from Inner surface to a desired surface, d. We are interested here in deriving the shell theory associated with the case where the axial and circumferential length scales are both equal to the inner radius of the cylinder, a, as follows:

$$L = a\ , \qquad l = a \tag{21}$$

The reason for taking the length scales **a** is the longest practical dimension of the shell and we are interested in developing membrane type theory which requires longer than the bending characteristic influential length according to the classical theory of isotropic materials, References (6) through (8) and (17) through (19).

On substituting these length scales into the three-dimensional elasticity equations (12) and (13) and stress-displacement relations and equilibrium equations of (14) and (15), we obtain:

If the asymptotic expansions (19) and (20) for the displacements and stresses are now substituted into equations (22) and (23), the following equations representing the first approximation theory of the problem result upon use of the procedure outlined in the last chapter. Note that both sides 1/2 of each equation are equated in like powers of λ and the leading terms may not correspond to $m=0$ term.

$$\begin{aligned} v_{r,y}^{(0)} &= 0 \\ v_{z,y}^{(0)} &= 0 \\ v_{\theta,y}^{(0)} &= 0 \end{aligned} \tag{22}$$

$$\begin{aligned} v_{z,x}^{(0)} &= s_{11}^{(0)} t_z^{(0)} + s_{12}^{(0)} t_\theta^{(0)} + s_{16}^{(0)} t_{\theta z}^{(0)} \\ v_{\theta,\phi}^{(0)} + v_r^{(0)} &= s_{21}^{(0)} t_z^{(0)} + s_{22}^{(0)} t_\theta^{(0)} + s_{26}^{(0)} t_{\theta z}^{(0)} \\ v_{\theta,x}^{(0)} + v_{z,\phi}^{(0)} &= s_{61}^{(0)} t_z^{(0)} + s_{62}^{(0)} t_\theta^{(0)} + s_{66}^{(0)} t_{\theta z}^{(0)} \\ t_{rz,y}^{(2)} + t_{\theta z,\phi}^{(0)} + t_{z,x}^{(0)} &= 0 \\ t_{r\theta,y}^{(2)} + t_{\theta,\phi}^{(0)} + t_{\theta z,x}^{(0)} &= 0 \\ t_{r,y}^{(2)} + t_\theta^{(0)} &= 0 \end{aligned} \tag{23}$$

The superscripts indicate the leading term in each of the expansions (18) and represent the relative of magnitude of the displacements and stresses. These orders of magnitude result from the intention to obtain a system of equations which is integrable with respect to the thickness coordinate y in a step-by-step manner and the following additional reasoning:

a) The dominant stress state in thin shell theory is the in-plane stress state. These stresses should be of the same order of magnitude.

b) The order of the displacements is chosen so that the product of the in-plane strains and the elastic moduli is of the same order of magnitude as the in-plane stresses.

c) The choice for the transverse stresses arises from the fact that they should contribute terms of the same magnitude in the equilibrium. Integration of the first three equations of (23) with respect to y yields

$$\begin{aligned} v_r^{(0)} &= v_r^{(0)}(x,\phi) \\ v_z^{(0)} &= v_z^{(0)}(x,\phi) \\ v_\theta^{(0)} &= v_\theta^{(0)}(x,\phi) \end{aligned} \tag{24}$$

where v_r, v_z , v_θ are the displacements of the $y=0$ ($r=a$) surface.

The middle three equations of (23) can be solved for the in-plane stresses as follows:

$$\begin{Bmatrix} t_z^{(0)} \\ t_\theta^{(0)} \\ t_{\theta z}^{(0)} \end{Bmatrix} = [C] \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} \tag{25}$$

Here, C. (i, j = 1, 2, 3) are the components of a symmetric matrix given by

$$C = \begin{bmatrix} s_{11}^{(0)} & s_{12}^{(0)} & s_{16}^{(0)} \\ s_{12}^{(0)} & s_{22}^{(0)} & s_{26}^{(0)} \\ s_{16}^{(0)} & s_{26}^{(0)} & s_{66}^{(0)} \end{bmatrix}^{-1} \tag{26}$$

and ε_1, ε_2 , ε_{12} are the in-plane strain components of the $y=0$ surface:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} = \begin{Bmatrix} v_{z,x}^{(0)} \\ v_{\theta,\phi}^{(0)} + v_r^{(0)} \\ v_{\theta,x}^{(0)} + v_{r,\phi}^{(0)} \end{Bmatrix} \tag{27}$$

On substituting the first approximation in-plane stress-strain relations (25) into the last three equations of (23) and integrating with respect to y, we obtain:

$$\begin{aligned} t_{rz} &= T_{rz}(x,\phi) - [A_{13}V_{z,x\phi} + A_{23}(V_{\theta,\phi\phi} + V_{r,\phi}) + A_{33}(V_{\theta,x\phi} + V_{z,\phi\phi})] \\ &\quad - [A_{11}V_{z,xx} + A_{12}(V_{\theta,x\phi} + V_{r,x}) + A_{13}(V_{\theta,xx} + V_{z,x\phi})] \\ t_{r\theta} &= T_{r\theta}(x,\phi) - [A_{12}V_{z,x\phi} + A_{22}(V_{\theta,\phi\phi} + V_{r,\phi}) + A_{23}(V_{\theta,x\phi} + V_{z,\phi\phi})] \\ &\quad - [A_{13}V_{z,xx} + A_{23}(V_{\theta,x\phi} + V_{r,x}) + A_{33}(V_{\theta,xx} + V_{z,x\phi})] \\ t_r &= T_r + A_{12}V_{z,x} + A_{22}(V_{\theta,\phi} + V_r) + A_{23}(V_{\theta,x} + V_{z,\phi}) \end{aligned} \tag{28}$$

where t_{rz}, $T_{r\theta}$, T_r are the transverse stress components of the $y=0$ surface and

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$$A_{ij} = \int_0^y C_{ij} \; d\rho \qquad (29)$$

In relations (28) and in what is to follow, the superscripts on the displacements have been dropped. Boundary conditions (8) are to be satisfied by each term of asymptotic expansions (18). This yields

$$\begin{aligned} &t_{rz}^{(2)} = t_{r\theta}^{(2)} = t_r^{(2)} = 0 \qquad (y=0) \\ &t_{rz}^{(2)} = t_{r\theta}^{(2)} = 0 \;, \quad t_r^{(2)} = p^* \quad (y=0) \end{aligned} \qquad (30)$$

Here, p^* is a dimensionless pressure defined by

$$p^* = p/(\sigma\lambda) \qquad (31)$$

Satisfaction of conditions (30) by (29) yields

$$T_{rz} = T_{r\theta} = T_r = 0 \qquad (32)$$

and the following three differential equations for displacements V_r , V_z and V_θ

$$\begin{aligned} &\underline{A}_{13}V_{z,x\phi} + \underline{A}_{23}(V_{\theta,\phi\phi} + V_{r,\phi}) + \underline{A}_{33}(V_{\theta,x\phi} + V_{z,\phi\phi}) \\ &+\underline{A}_{11}V_{z,xx} + \underline{A}_{12}(V_{\theta,x\phi} + V_{r,x}) + \underline{A}_{13}(V_{\theta,xx} + V_{z,x\phi}) = 0 \\ &\underline{A}_{12}V_{z,x\phi} + \underline{A}_{22}(V_{\theta,\phi\phi} + V_{r,\phi}) + \underline{A}_{23}(V_{\theta,x\phi} + V_{z,\phi\phi}) \\ &+\underline{A}_{13}V_{z,xx} + \underline{A}_{23}(V_{\theta,x\phi} + V_{r,x}) + \underline{A}_{33}(V_{\theta,xx} + V_{z,x\phi}) = 0 \\ &\underline{A}_{12}V_{z,x} + \underline{A}_{22}(V_{\theta,\phi} + V_r) + \underline{A}_{23}(V_{\theta,x} + V_{z,\phi}) = p^* \end{aligned} \qquad (33)$$

In the above equations

$$\underline{A}_{ij} = A_{ij}^{(1)} \qquad (34)$$

To obtain the appropriate expressions for the stress resultants we first non-dimensionalize those defined by (9) as follows:

$$\bar{N} = \frac{N}{\sigma \cdot \lambda \cdot \mathrm{a}} \;, \quad \bar{M} = \frac{M}{\sigma \cdot \lambda^2 \cdot \mathrm{a}^2} \qquad (35)$$

where N and M are the generic symbol for the force and moment stress resultants, respectively. Assuming it to be possible, we now asymptotically expand each of the dimensionless stress resultants in a power series in 1/2,

$$\bar{N} = \sum_{m=0}^{M} N^{(m)}(x,\phi)\cdot\lambda^{m/2}$$
$$\bar{M} = \sum_{m=0}^{M} M^{(m)}(x,\phi)\cdot\lambda^{m/2} \qquad (36)$$

where $N^{(m)}$ and $M^{(m)}$ are of the order unity.

Pseudo-Membrane Phenomena

We are now interested in a formulation of equations to be able to obtain all the stress resultants due to membrane and bending actions.
On substitution (35), (36) and the results for in-plane stresses (24) into relations (9) and equating terms of like powers in 1/2 on each side of the equations, we obtain the following expressions for the first approximation stress resultants:

$$\begin{Bmatrix} N_z \\ N_\theta \\ N_{z\theta} \\ N_{\theta z} \\ M_z \\ M_\theta \\ M_{z\theta} \\ M_{\theta z} \end{Bmatrix} = \begin{bmatrix} \bar{A} \\ \bar{\underline{B}} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_{12} \end{Bmatrix} \qquad (37)$$

where the superscript zero have been omitted and B. is defined as follows:

$$B_{ij} = \int_0^y C_{ij}(\rho)\, d\rho \ , \qquad \underline{B}_{ij} = B_{ij}(1) \qquad (38)$$

and submatrices $\left[\bar{A}\right]$ and $\left[\bar{B}\right]$ are given by

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$$[\bar{A}] = \begin{bmatrix} \frac{1}{1+d/a}[\underline{A}_{11}, \underline{A}_{12}, \underline{A}_{13}] \\ \underline{A}_{21}, \underline{A}_{22}, \underline{A}_{23} \\ \frac{1}{1+d/a}[\underline{A}_{31}, \underline{A}_{32}, \underline{A}_{33}] \\ \underline{A}_{31}, \underline{A}_{32}, \underline{A}_{33} \end{bmatrix}$$

$$[\bar{B}] = \begin{bmatrix} \frac{1}{1+d/a}\left[\left(-\frac{d}{h}\underline{A}_{11} + \underline{B}_{11}\right), \left(-\frac{d}{h}\underline{A}_{12} + \underline{B}_{12}\right), \left(-\frac{d}{h}\underline{A}_{13} + \underline{B}_{13}\right)\right] \\ \left(-\frac{d}{h}\underline{A}_{21} + \underline{B}_{21}\right), \left(-\frac{d}{h}\underline{A}_{22} + \underline{B}_{22}\right), \left(-\frac{d}{h}\underline{A}_{23} + \underline{B}_{23}\right) \\ \frac{1}{1+d/a}\left[\left(-\frac{d}{h}\underline{A}_{31} + \underline{B}_{31}\right), \left(-\frac{d}{h}\underline{A}_{32} + \underline{B}_{32}\right), \left(-\frac{d}{h}\underline{A}_{33} + \underline{B}_{33}\right)\right] \\ \left(-\frac{d}{h}\underline{A}_{31} + \underline{B}_{31}\right), \left(-\frac{d}{h}\underline{A}_{32} + \underline{B}_{32}\right), \left(-\frac{d}{h}\underline{A}_{33} + \underline{B}_{33}\right) \end{bmatrix}$$

(39)

Note that d/a can be written as

$$d/a = \lambda(d/h) \tag{40}$$

From the results obtained above, we characterize the theory as follows:

a) The approach that this research took, the asymptotic integration, for deriving shell equations is capable of obtaining all stress components, including the transverse components.

b) The first three equations of (22) result from the relations for the transverse strains. The variation with respect to y is zero as shown in the displacements (23) which are independent of y. The strain components of any point y off the $y=0$ surface are thus equal to those of the $y=0$ surface, similar to classical membrane theory.

c) The stress components vary with y because as the C_{ij}, and the A_{ij}. are functions of y.

d) Equations (37) show that moment stress resultants are produced due to the non-homogeneity of the material.

For an isotropic and homogeneous material, the C_{ij} are constants and d/h = 1/2. This yields

$$\underline{B}_{ij} = \frac{1}{2}C_{ij} = \frac{1}{2}\underline{A}_{ij} \tag{41}$$

On substituting this result into relations (38) it is seen that submatrix B is equal to zero and that relations (37) become those of the classical membrane theory of shell (zero moment resultants). In case of hybrid anisotropic materials, it is very rare to satisfy all the components of the submatrix $[B]$ to be equal to zero at the same time. Another way of observation, it is

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unavoidable to associate with some bending moments in addition to pure membrane forces for laminated anisotropic shell walls. Therefore, the analysis is named pseudo-membrane theory. It is different from the long effective length of Vlasov's semi-membrane theory nor the short effective length of Donnell's theory.

Application

To demonstrate the validity of the theory developed here, we will choose a problem of a laminated circular cylindrical shell under internal pressure and edge loadings. The shell is assumed to build with boron/epoxy composite layers. Each layer is taken to be taken to be homogeneous but anisotropic with an arbitrary orientation of the elastic axes. We need not consider the restriction of the symmetry of the layering due to the non-homogeneity considered in the original development of the theory expressed by equation (5). Thus each layer can possess a different thickness.

We assume here that the contact between layers is such that the strains are continuous function in thickness coordinate. As the C_{ij} are piecewise continuous functions, the in-plane stresses are also continuous. We would expect them to be discontinuous at the juncture of layers of dissimilar materials. The transverse stresses are continuous functions of the thickness coordinate.

Although as mentioned above the theory developed can take unlimited hybrid random layers but for an example, a four layer symmetric angle ply configuration. For this configuration the angle of elastic axes γ is oriented at $+\gamma$, $-\gamma$, $-\gamma$, $+\gamma$ wIth the shell axis and the layers are of equal thickness.

Let the shell be subjected to an internal pressure p, an axial force per unit circumferential length N. The axial force is taken to be applied at $r = a + H$ such that a moment $N(H - d)$ is produced about the reference surface $r = a + d$. We introduce dimensionless external force and moments as follows:

$$\bar{N} = \frac{N}{\sigma \cdot \lambda \cdot \alpha}$$
$$\bar{M} = \frac{N(H-d)}{\sigma \cdot \lambda^2 \cdot \alpha^2} \qquad (42)$$
$$\bar{T} = \frac{T}{2\pi\sigma \cdot \lambda^2 \cdot \alpha^3 (1 + d/a)}$$

To demonstrate the validity of the derived theory, we have simplified loading and boundary conditions as follows:

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$$
\begin{aligned}
&V_r = V_{r,x} = V_z = V_\theta && (x=0,\ \ y=d/h) \\
&v_r = 0,\ \bar{N}_2 = N,\ \ \bar{M}_2 = M && (x=l,\ \ y=d/h) \qquad (43) \\
&(1+d/a)\bar{N}_{z\theta} + \lambda \bar{M}_{z\theta} = \bar{T}
\end{aligned}
$$

Here, l is the dimensionless length of the cylindrical shell.

In the theories developed in the previous chapters, the distance d at which the stress resultants were defined was left arbitrary. We now choose it to be such that there exists no coupling between N_z and K_1 and M_z and C_{ij} .
As the loading applied at the end of the shell is axi-symmetric, all the stresses and strains are also taken to be axi-symmetric. We thus can set all the derivatives in the expressions for the stresses and strains and in the equations for the displacements equal to zero.

Numerical calculations are now carried out for a shell of wall of various hybrid laminae.
Each of the layers is taken to be equal thickness and thus the dimensionless distances from the bottom of the first layer are given by

$$S_1 = 0,\ S_2 = 0.25,\ S_3 = 0.5,\ S_4 = 0.75,\ S_5 = 1.0$$

each layer of the symmetric angle ply configuration (elastic symmetry axes y are oriented at $(+\gamma$, $-\gamma$, $-\gamma$, $+\gamma)$ is taken to be orthotropic with engineering elastic coefficients representing those for a boron/epoxy material system,

$$
\begin{aligned}
&E_1 = 2.413\times 10^5 \text{ MPa} \\
&E_2 = 1.0\times 10^5 \text{ MPa} \\
&G_{12} = 5.17\times 10^5 \text{ MPa}
\end{aligned}
$$

Here direction 1 signifies the direction parallel to the fibers while 2 is the transverse direction. Angles chosen were $\gamma = 0$, 15, 30, 45 and 60. Use of the transformation equations (2.6) then yields the mechanical properties for the different symmetric angle ply configurations.
We next apply the following edge loads: $N = p$ and take $\sigma = p/\lambda$, $H = (3/4)h$ and the reference surface we take $d/h = 1/2$

Shown in Figs.4 to Fig. 6 is the variation of the dimensionless radial displacement with the actual distance along the axis for the different theories. The reference surface for the chosen configuration is given by $d/h = 1/2$. The integration constants determined from the edge conditions.

It is also seen that wide variations in the magnitude of radial displacement take place with change in the cross-ply angle. The maximum displacement occurs at γ = 30 degree while the minimum displacement is at γ = 60 degree. Because we have simplified all the conditions to be purely membrane status, membrane stress as well as displacements cannot accommodate with the edge conditions as shown in the Figure 5. Also shown in the Figure 6 is the patterns of near edge zone to compare the pure membrane theory against bending theory, which are close to Donnel's theory for the case of isotropic material. The results of bending theory were adopted from the Reference (12).

In each case, the displacements increase with increase in γ up to γ=30 degree and thereafter decrease.

CONCLUSION

In the present analysis, first approximation shell theories are derived by use of the method of asymptotic integration of the exact three-dimensional elasticity equations for a non-homogeneous anisotropic circular cylindrical shell. The analysis is valid for materials which are non-homogeneous to the extent that their properties are allowed to vary with the thickness coordinate (r).

The first approximation theory derived in this analysis represent the simplest possible shell theories for the corresponding length scales considered. Although twenty one elastic coefficients are present in the original formulation of the problem, only six are appear in the first approximation theories. It was seen that use of the asymptotic method employed in the research also yields expressions for all stress components, including the transverse ones. Unlike the pure membrane theory of isotropic materials, secondary bending moments can be computed in association of material characteristics of lamination.

The fact that these expressions can be determined is very useful when discussing the possible failure of composite shells and also for the discrepancy between theoretical membrane theory and experimental results.

For design of space shuttles and other vehicles, a shell structure must be carefully designed for all possible loading conditions, extremely high negative and positive pressure and temperature, which demands further accurate shell theories. In case the membrane theory seems to be justified, the effect of all possible secondary bending moments must carefully be examined as shown in the equation (37) through (41) of this analysis. It is more realistic for shells of hybrid anisotropic materials of high strength.

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The asymptotic integration method that applied in this formulation was first developed by Dr. O.E. Widera for his dynamic theory of shell structures, References (4) and (5), the authors concentrated on the development of the theories applied to pressure vessels and analyzed the behavior of shells of hybrid laminated composites.

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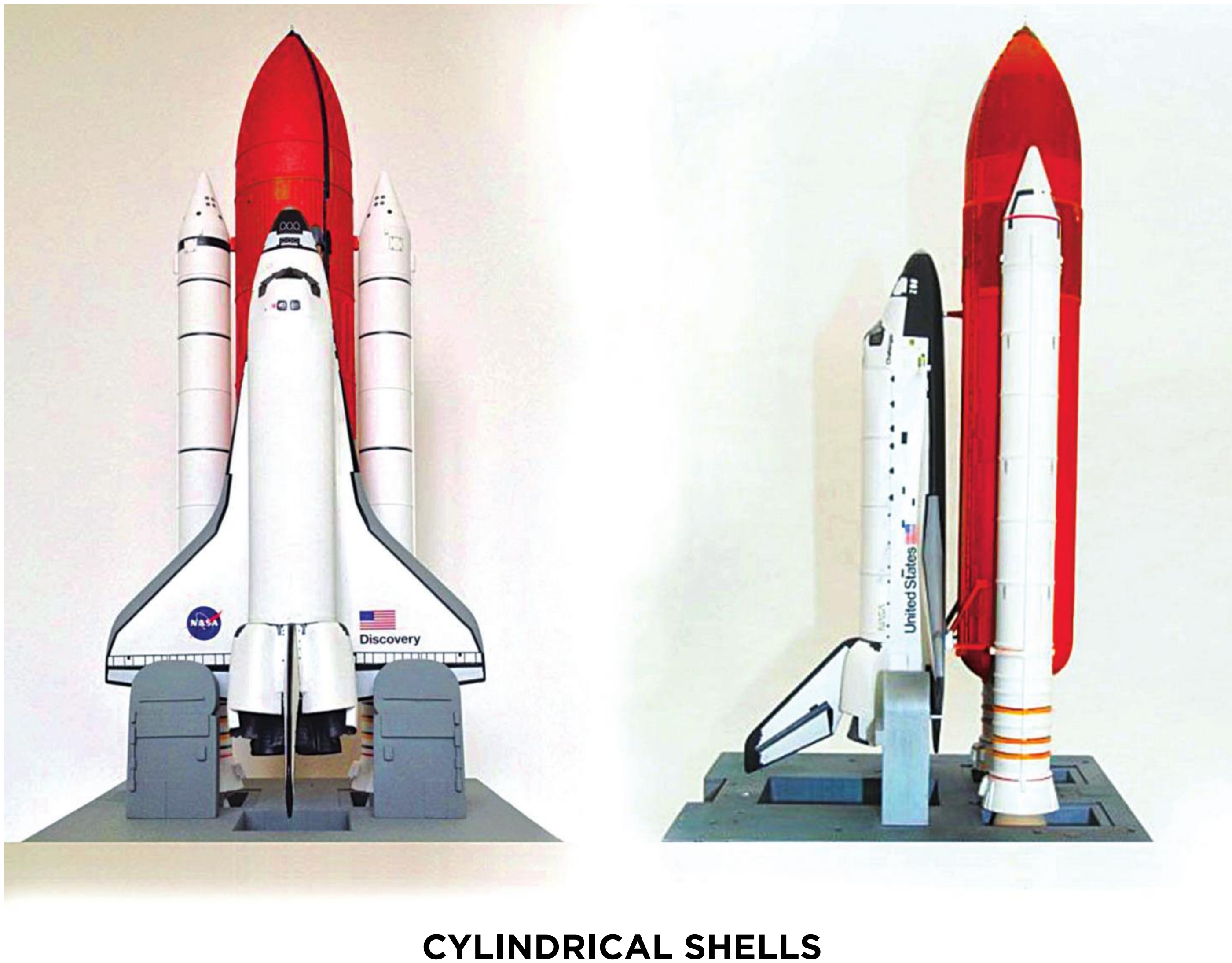

Figure 1, Examples of Cylindrical Shells

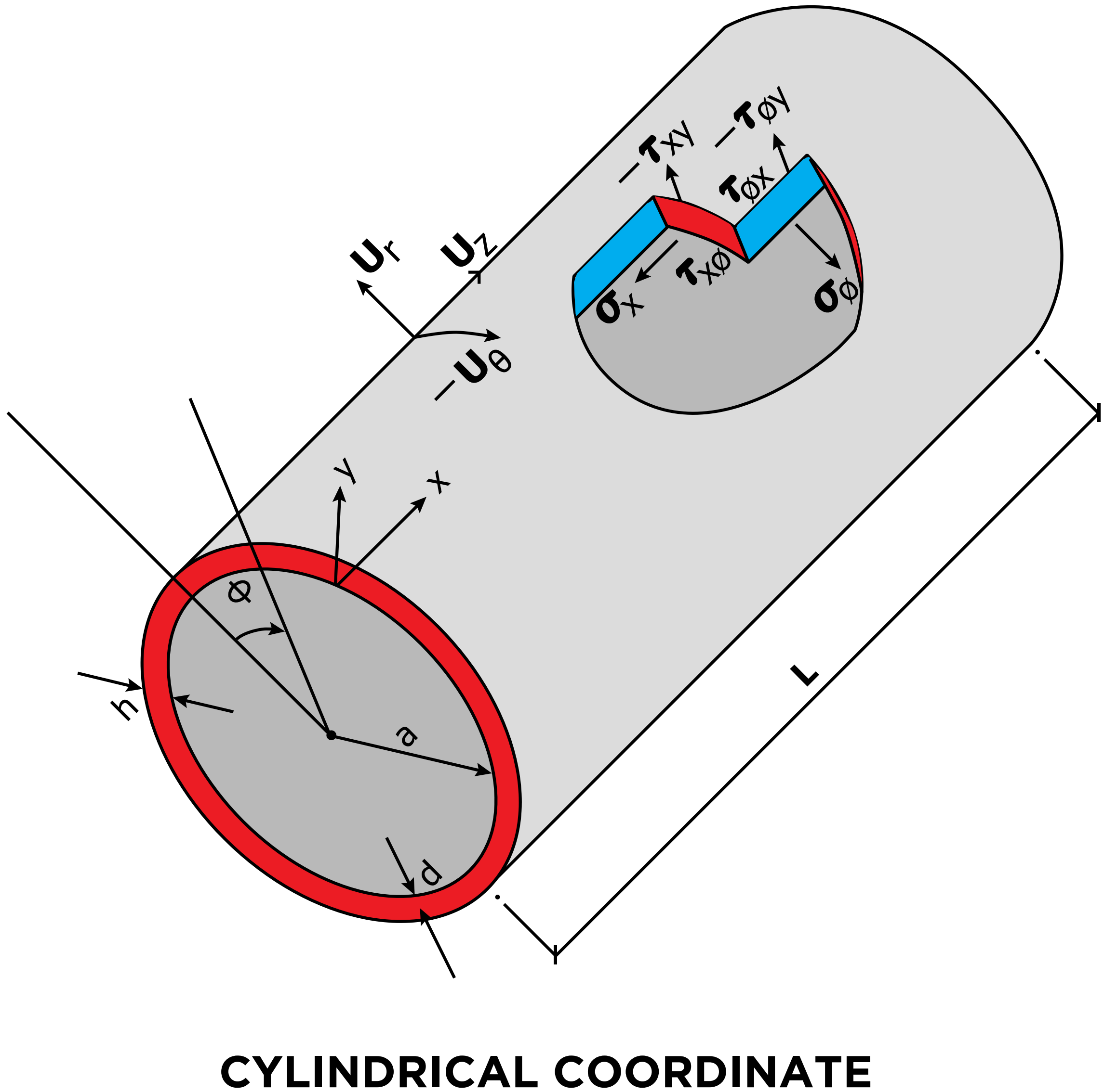

Figure 2. Dimensions, Deformations and Stresses of the Cylindrical Shell

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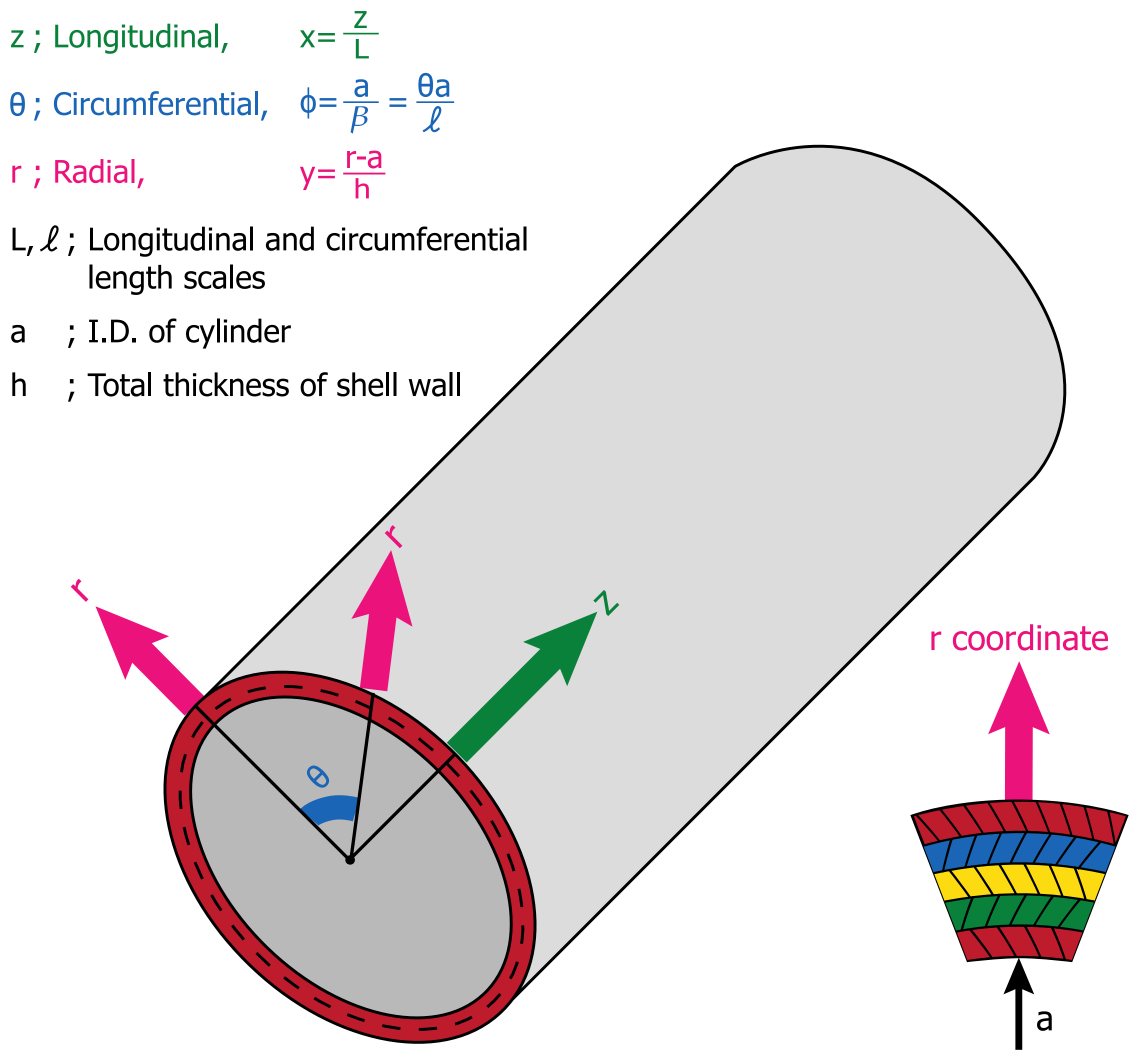

Figure 3, Details of the Coordinate System

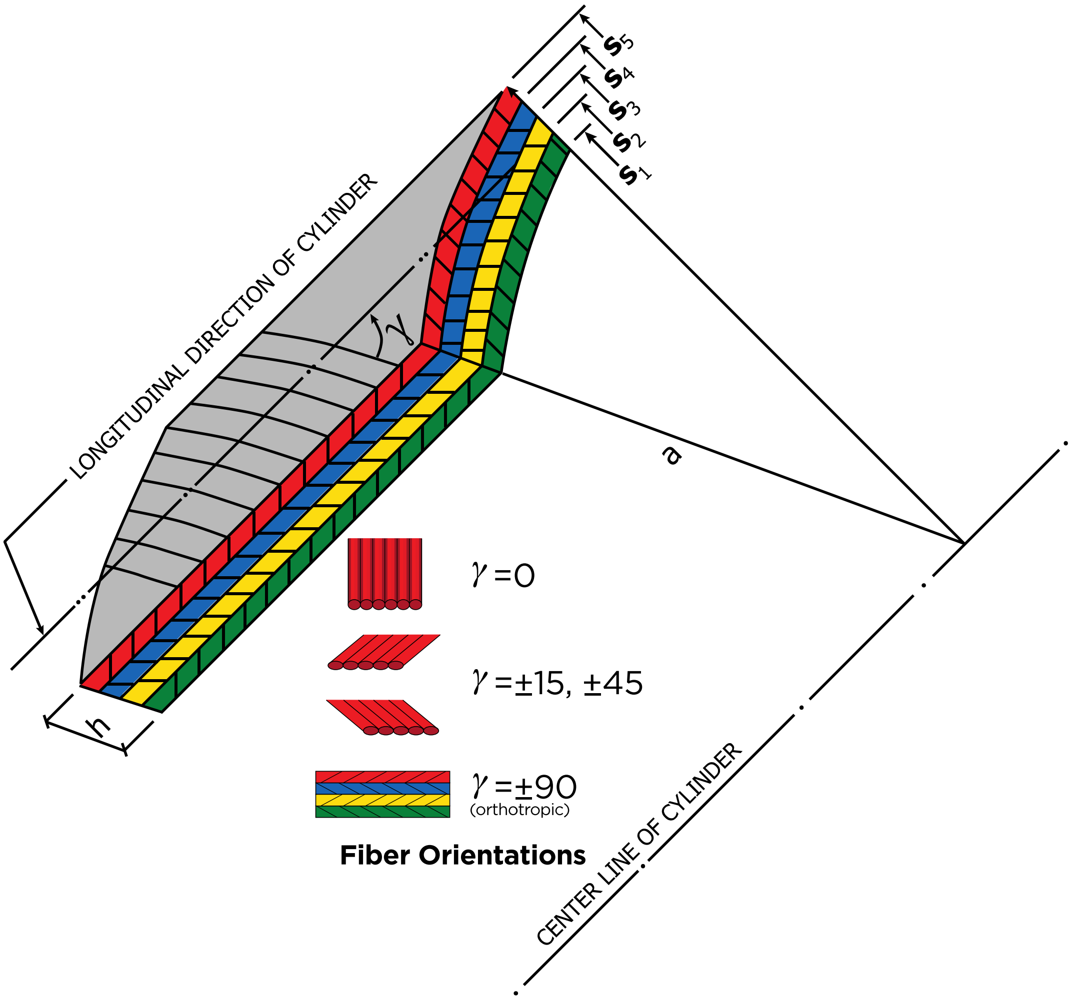

Figure 4, A Laminated Cylindrical Shell, Material Orientation γ

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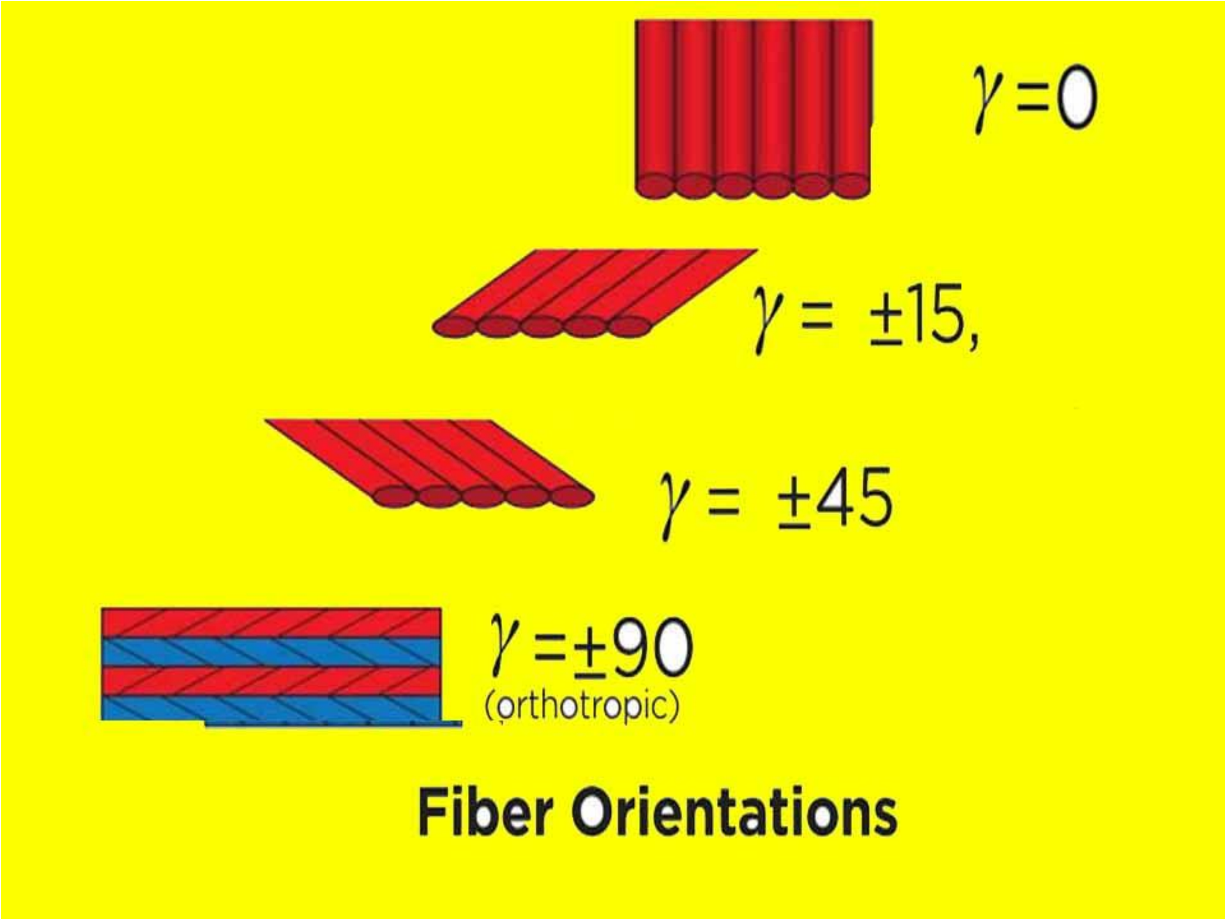

Figure 5, Fiber Orientations

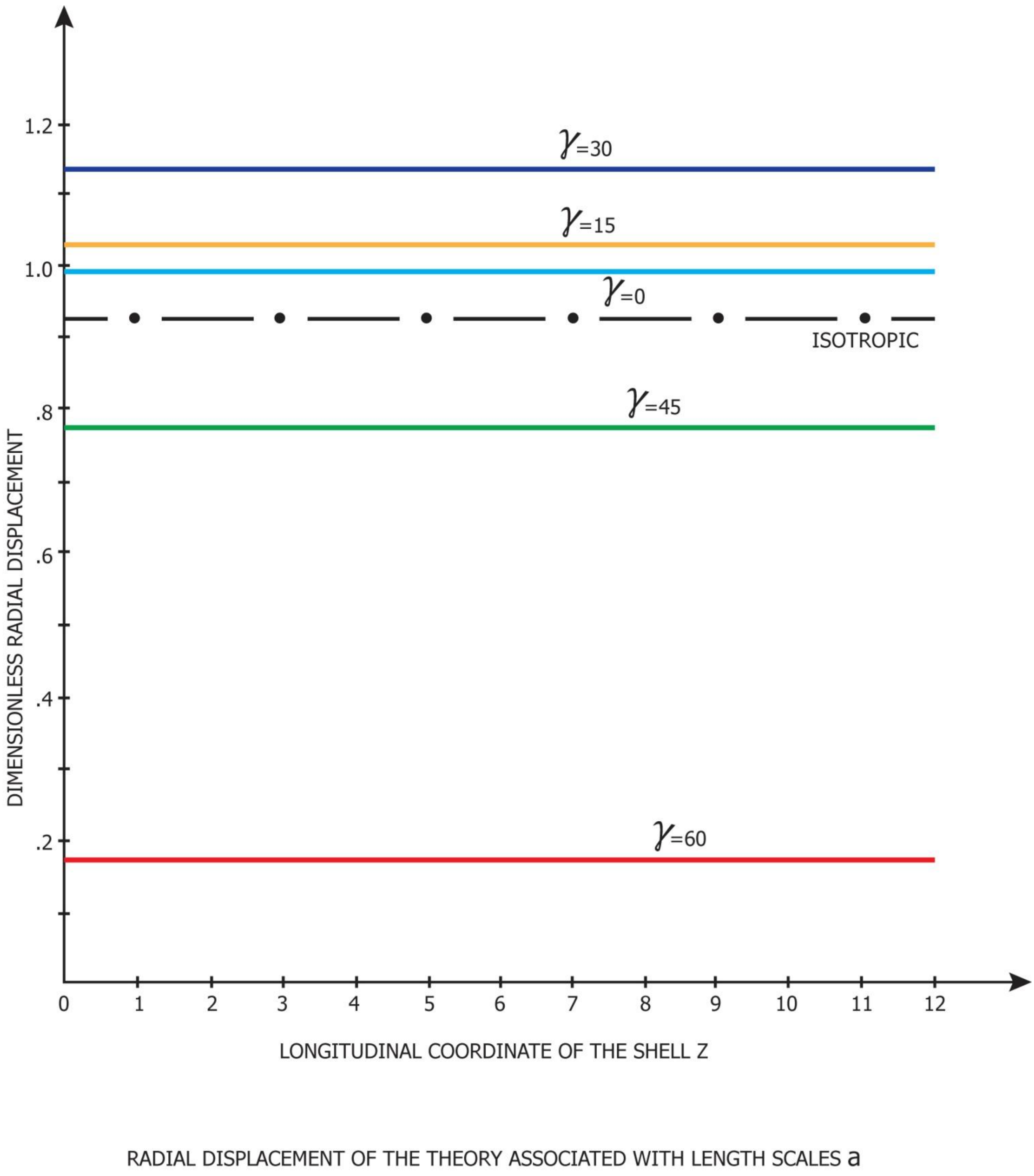

Figure 6, Non-dimensionalized Radial Displacement of Pure Membrane Theory

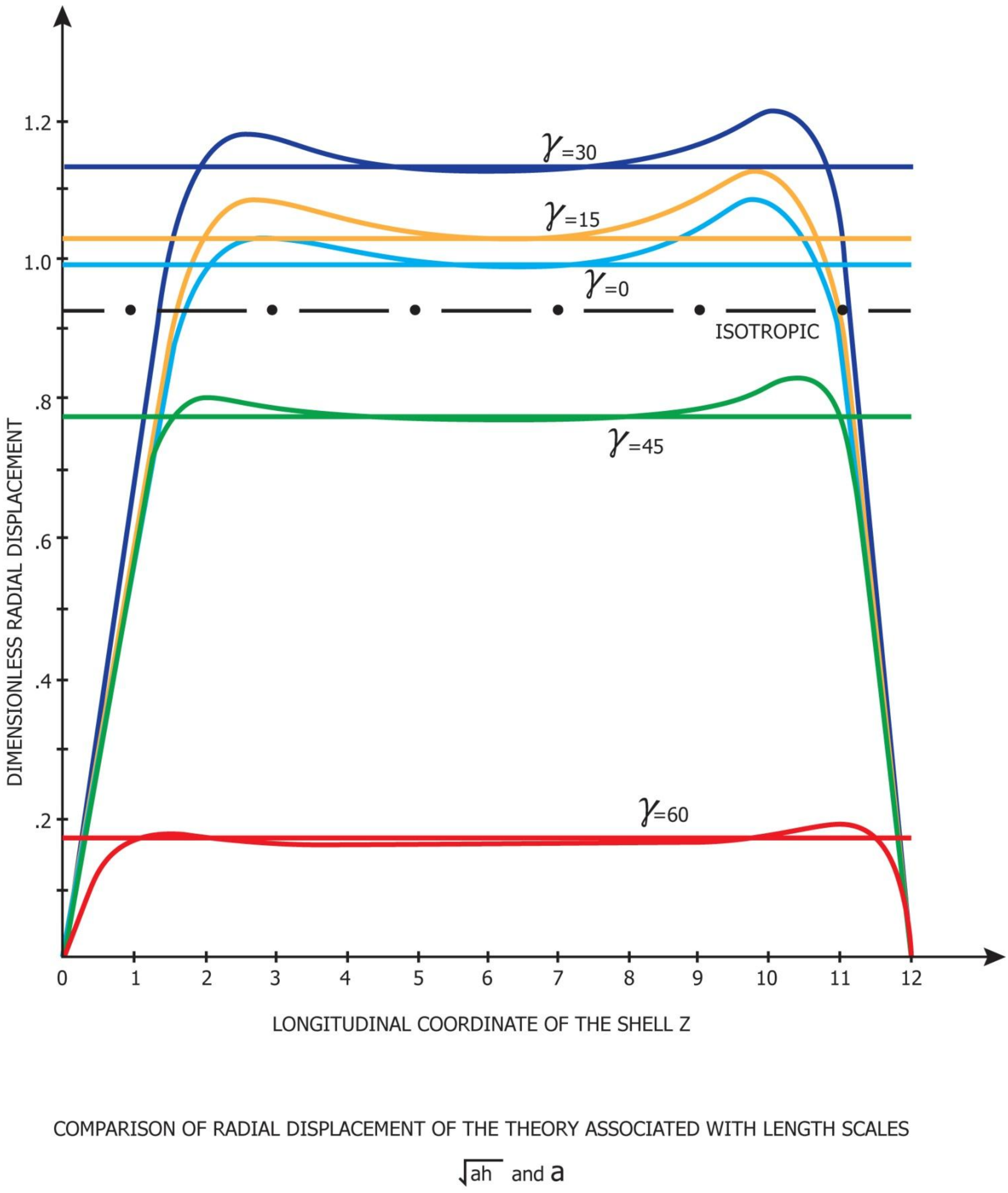

Figure 7, Comparison of Radial Displacements of the Bending and Pure Membrane Theories.