SOME CRITERIA FOR PLURIHARMONICITY

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ABSTRACT

In this article we prove an analogue of Hartogs' Lemma for pluriharmonic functions:

Theorem I. Suppose that the function $U({}'z, z_n)$ is defined in polydisk ${}'V \times \{|z_n| < R\} \subset C^{n-1}_{{}'z} \times C_{z_n}, R > 0$ and satisfies the following conditions:

1) For each fixed ${}'z \in {}'V$ function $U({}'z, z_n)$ of a complex variable z_n harmonic in the circle $|z_n| < R$

2) $U({}'z, 0)$ is harmonic in ${}'V$ on ${}'z$

3) function $U({}'z, z_n)$ is plurisubharmonic in all variables in some polydisk ${}'V \times \{|z_n| < r\},\ R > r > 0$

Then, the function $U({}'z, z_n)$ is pluriharmonic in polydisk ${}'V \times \{|z_n| < R\}$

Keywords: Hartogs' Theorem, Lelong's theorem, separately holomorphic functions, separately harmonic functions, holomorphic functions, harmonic functions, plurisubharmonic functions.

INTRODUCTION

Theorem of Hartogs and Lelong claims that separately holomorphic or separately-harmonic functions are, respectively, holomorphic and harmonic functions in all variables. The main difficulty in the proof of these theorems is to establish continuity (and even semi- continuity or limitations) considered functions in the variables. In this regard, the example of Yan Vigerinskiy is instructive, showing that there are separately-subharmonic functions $f(z_{1,} z_2)$, not limited at the top in all the variables, thus not being subharmonic in all the variables.

The proof of Hartogs' theorem is based on the following Hartogs' lemma: if the function $f({}'z, z_n)$ is holomorphic in polydisk and for each fixed ${}'z \in {}'V$ by Z_n holomorphic continues in a large circle $|z_n| < R, R > r$, it extends holomorphically according to the set of variables in the polydisk ${}'V \times \{|z_n| < R\}$. It is a fair analogue of this lemma for pluriharmonic functions: let the function $U({}'z, z_n)$ be pluriharmonic in polydisk ${}'V \times \{|z_n| < r\}, r > 0,$

and each fixed ${}^{/}z \in{}^{/} V$ harmonically extends for Zn in a big circle $|z_n| < R, R > r$. Then U is a large pluriharmonic in a big polydisk ${}^{/}V \times \{|z_n| < R\}$.

In fact, since U is pluriharmonic in ${}^{/}V \times \{|z_n| < r\}$, then U is a real part of a holomorphic function $U = \mathrm{Re}\, f$, f-holomorphic in ${}^{/}V \times \{|z_n| < r\}$. In addition, for fixed ${}^{/}z^0 \in{}^{/} V$, it is a real part of a holomorphic $|z_n| < R$ function

$F_{{}^{/}z^0}(z_n) : U({}^{/}z^0, z_n) = \mathrm{Re}\, F_{{}^{/}z^0}(z_n)$

Let's consider the difference $F_{{}^{/}z^0}(z_n) - f({}^{/}z^0, z_n)$. Since $\mathrm{Re}(F_{{}^{/}z^0}(z_n) - f({}^{/}z^0, z_n)) = 0$ is in $\{|z_n| < r\}$, from the uniqueness theorem it follows that $F_{{}^{/}z^0}(z_n) - f({}^{/}z^0, z_n) = iC({}^{/}z^0)$. Consequently, the function $f({}^{/}z^0, z_n) = F_{{}^{/}z^0}(z_n) - iC({}^{/}z^0)$ is holomorphic on z_n in a large circle $|z_n| < R$, for each fixed ${}^{/}z \in{}^{/} V$ and holomorphic in a smaller semicircle ${}^{/}V \times \{|z_n| < R\}$.

Then, applying Lemma Hartogs, we get f holomorphicity in polydisk ${}^{/}V \times \{|z_n| < R\}$ and, therefore, the U is pluriharmonic in this circle.

This article discusses the issues of pluriharmonic functions' extension along the axis direction OZ_n . the pluriharmonic continuation of the functions is proven in the direction of Theorem I.

An analogue of Hartogs' Lemma for Pluriharmonic Functions

Theorem I. Suppose that the function $U({}^{/}z, z_n)$ is defined in polydisk ${}^{/}V \times \{|z_n| < R\} \subset C^{n-1}_{{}^{/}z} \times C_{z_n}, R > 0$ and satisfies the following conditions:

4) For each fixed ${}^{/}z \in{}^{/} V$ the function $U({}^{/}z, z_n)$ of a complex variable is harmonic in the circle $|z_n| < R$.

5) $U({}^{/}z, 0)$ is harmonic in ${}^{/}V$ on ${}^{/}z$.

6) The function $U({}^{/}z, z_n)$ is plurisubharmonic in all variables in a polydisk ${}^{/}V \times \{|z_n| < r\},\ R > r > 0$

Then the function $U({}^{/}z, z_n)$ is pluriharmonic in the polydisk ${}^{/}V \times \{|z_n| < R\}$.

To prove Theorem I, we need the following lemma, which is a harmonic analogue of this theorem.

Lemma I. Let the function $U({}^{/}z, z_n)$ is defined in the polydisk ${}^{/}V \times V_n = {}^{/}V \times \{|z_n| < R\} \subset C^{n-1}_{{}^{/}z} \times C_{z_n}, R > 0$ and satisfies the following conditions:

1) For each fixed ${}^{/}z \in {}^{/}V$, the function $U({}^{/}z, z_n)$ of the variable z_n is harmonic in the circle V_n.
2) The function $U({}^{/}z, 0)$ is harmonic in ${}^{/}U$ on ${}^{/}z$.
3) 3) The function $U({}^{/}z, z_n)$ is subharmonic in the polydisk ${}^{/}V \times V_n$.

Then the function $U({}^{/}z, z_n)$ is harmonic in polydisk ${}^{/}V \times V_n$.

Evidence. We take an arbitrary number $r < R$ and for fixed ${}^{/}z \in {}^{/}V$ and $|z_n| < r$ according to Poisson's formula we express $U({}^{/}z, z_n)$ in terms of an integral over the circle,

$$U({}^{/}z, z_n) = \frac{1}{2\pi} \int_0^{2\pi} U({}^{/}z, \xi) \operatorname{Re}\left(\frac{\xi + z_n}{\xi - z_n}\right) dt \qquad \text{(I)}$$

where $\xi = re^{it}$.

First, for the clarity of presenting, we consider the case, when $U \in C^2\left({}^{/}V \times V_n\right)$, that is, U is a twice smooth function of the variables.

According to terms of lemma *Laplace* operator

$$\Delta U = \Delta_{{}^{/}z} U + \Delta_{z_n} U = \Delta_z U \geq 0$$

Since U is harmonic on z_n, and hence $\Delta_{z_n} U = 0$ for any fixed ${}^{/}z \in {}^{/}V$. From this and from formula (I) we have $\Delta U = \Delta_{{}^{/}z} U = \frac{1}{2\pi} \int_0^{2\pi} \Delta_{{}^{/}z} U({}^{/}z, \xi) \operatorname{Re}\left(\frac{\xi + z_n}{\xi - z_n}\right) dt \geq 0$.

Let's consider the function

$\psi({}^{/}z, z_n) = \frac{1}{2\pi} \int_0^{2\pi} \Delta_{{}^{/}z} U({}^{/}z, \xi) \frac{\xi + z_n}{\xi - z_n} dt$ the real part of which matches ΔU. It is clear that it is holomorphic according to z_n in the circle ${}^{/}V \times \{|z_n| < r\}$ for any fixed ${}^{/}z \in {}^{/}V$. Now, the condition 3 of Lemma implies that the Laplace operator $\Delta_{{}^{/}z} U({}^{/}z, 0) = 0$.

According to the mean value theorem (which also follows from the expression (I) when $z_n = 0$) we have

$$\Delta_{{}^{/}z}U({}^{/}z,0)=\frac{1}{2\pi}\int_{0}^{2\pi}\Delta_{{}^{/}z}U({}^{/}z,\xi)dt$$

Thence_ $\psi({}^{/}z,0)=\frac{1}{2\pi}\int_{0}^{2\pi}\Delta_{{}^{/}z}U({}^{/}z,\xi)dt=\Delta_{{}^{/}z}U({}^{/}z,0)=0$.

From the property of holomorphic functions' openness, the holomorphic function on z_n $\psi({}^{/}z,z_n)$ either $\equiv 0$ or sends a neighborhood of zero in a neighborhood of zero. This is impossible due to the fact that $\text{Re}\,\psi=\Delta U\geq 0$ in ${}^{/}V\times\{|z_n|<r\}$.

Consequently, $\psi\equiv 0$. Thus $\Delta U=0$ is in ${}^{/}V\times\{|z_n|<r\}$. Since $r<R$ is arbitrary, from this we obtain the harmony of the function $U({}^{/}z,z_n)$ in ${}^{/}V\times V_n$.

Now let's consider the general case where U is an arbitrary subharmonic function.

According to the condition of the lemma in this case, the Laplacian ΔU is positive in the generalized sense of functionality defined by the integral $(\Delta U,\varphi)=(U,\Delta\psi)=\int\Delta U\varphi dV$

where φ is the infinitely smooth, compactly supported in polydisk ${}^{/}V\times V_n$ function, is positive, ie, $(\Delta U,\varphi)\geq 0$ when $\varphi\geq 0$. In particular, for any finite non-negative functions $\varphi_1({}^{/}z)$,$\varphi_2(z_n)$ of the class C^{∞}, and respectively in the domains ${}^{/}V$ and V_n, we have

$$(U,\Delta(\varphi_1\cdot\varphi_2))=\int_{{}^{/}V\times V_n}U({}^{/}z,z_n)\Delta(\varphi_1({}^{/}z),\varphi_2(z_n))dV\geq 0$$

where dV are the forms of the volume in C^n

As $\Delta(\varphi_1\cdot\varphi_2)=(\Delta_{{}^{/}z}\varphi_1)\varphi_2+\varphi_1(\Delta z_n\varphi_2)$.

According to Fubini's theorem and from the harmony U on z_n we have

$$(U,\Delta(\varphi_1\cdot\varphi_2))=\int_{V_n}\left\{\int_{{}^{/}V}U\Delta_{{}^{/}z}\varphi_1 d^{/}V\right\}\varphi_2 dV_n+\int_{V_n}\left\{\int_{{}^{/}V}U\Delta_{{}^{/}z}\varphi_2 dV_n\right\}\varphi_1 d^{/}V=$$

$$=\int_{V_n}\left\{\int_{{}^{/}V}U\Delta_{{}^{/}z}\varphi_1 d^{/}V\right\}\varphi_2 dV_n$$

where $d^{/}V$ and dV_n are forms of the volume in $C^{n-1}_{{}^{/}z}$ and C_{z_n} respectively.

Therefore,

$$\left(U,\Delta\left(\varphi_1\cdot\varphi_2\right)\right)=\int_{V_n}\left\{\int_{^/V}\Delta\varphi_1 d^/V\right\}\varphi_2 dV_n\geq 0$$

For all finite non-negative functions $\varphi_2(z_n)$
Consequently, the inner integral is positive:

$$\int_{^/V}U(^/z,z_n)\Delta\varphi_1(^/z)d^/V$$

For each fixed $z_n\in V_n$

Hence, according to Fubini's theorem and the formula (I) for each fixed $r<R$ and for each fixed $z_n\in V_n^/=\{|z_n|<r\}$
We have

$$(U,\Delta\varphi_1)=\int_{^/V}\left\{\frac{1}{2\pi}\int_0^{2\pi}U(^/z,\xi)\operatorname{Re}(\frac{\xi+z_n}{\xi-z_n})dt\right\}\Delta\varphi_1 d^/V=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{^/V}U(^/z,\xi)\Delta\varphi_1 d^/V\right\}\operatorname{Re}(\frac{\xi+z_n}{\xi-z_n})dt\geq 0$$

Where $\xi=re^{it}$

It is clear that the function $\psi(z_n)=\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{^/V}U(^/z,\xi)\Delta\varphi_1 d^/V\right\}\frac{\xi+z_n}{\xi-z_n}dt$ is holomorphic on z_n in $V_n^/$, and

$$\psi(0)=\frac{1}{2\pi}\int_0^{2\pi}\left\{\int_{^/V}U(^/z,\xi)\Delta\varphi_1 d^/V\right\}dt=\int_{^/V}U(^/z,0)\Delta\varphi_1(^/z)d^/V=0$$

According to the mean value theorem and the condition of lemma.
As

$(U,\Delta\varphi_1)=\operatorname{Re}\psi\geq 0$ in $V_n^/$, it follows that $\psi\equiv 0$.
Hence,

$$\int_{^/V}U(^/z,z_n)\Delta\varphi_1 d^/V=0$$

For each fixed $|z_n|<r$ and for any non-negative in $^/V$ function φ_1. Consequently, for each fixed $z_n\in V_n$ the function $U(^/z,z_n)$ is harmonic in $^/V$ on $^/z$. Now it

is easy to show that $\Delta U = 0$ in a generalized sense. Indeed, for any finite in polydisk ${}^{/}V \times V_n$ function ψ we have:

$$(U,\Delta\psi) = \int U \Delta \psi dV = \int U(\Delta_{{}^{/}z}\psi + \Delta_{z_n}\psi) dV =$$

$$= \int U \Delta_{{}^{/}z} \psi dV + \int U \Delta_{z_n} \psi dV = 0$$

Hence, the Laplace operator ΔU from a subharmonic function U is zero in the generalized sense.
Such subharmonic function must be submitted in accordance with the harmonic Riesz. The proof is complete.
We turn to the proof of the theorem. It follows from the proof of lemma and the following elementary fact: if the function U is simultaneously harmonic and pluriharmonic in polydisk ${}^{/}V \times V_n$, it is pluriharmonic in the polydisk ${}^{/}V \times V_n$.
From the condition of Theorem I, according to Lemma I, it follows that $U({}^{/}z, z_n)$ is harmonic in the polydisk ${}^{/}V \times \{|z_n| < r\}$.
Then, the plurisubharmony U in this polydisk implies that the function U is **pluharmonic** in ${}^{/}V \times \{|z_n| < r\}$. From the analogue of Hartogs' Lemma for **pluharmonic** functions (see the beginning of Chapter 1), we find that U is **pluharmonic in** a large polydisk ${}^{/}V \times \{|z_n| < R\}$.
The theorem has been proven.

Note using the result of E.M. Cirka, formulated in the introduction, it is possible to strengthen Lemma I, by demanding in condition 3 of Lemma I. Subharmonicity of the function not in the whole polydisk ${}^{/}V \times V_n$, and in some smaller polydisk ${}^{/}V \times \{|z_n| < r\},\ r < R.$
Indeed, from proven lemma I, in this case we obtain harmony U and, consequently, the real analytic U in polydisk ${}^{/}V \times \{|z_n| < r\}$, with a fixed ${}^{/}z_0 \in {}^{/}V$ function $U({}^{/}z^0, z_n)$ of the variable z_n will be harmonic in the circle $|z_n| < R$. Theorem E.M. Cirka implies that U real analytic in a large polydisk ${}^{/}V \times V_n$ .In addition, the Laplace operator will be the real-analytic ${}^{/}V \times V_n$.
Since $\Delta U \equiv 0$ in the polydisk ${}^{/}V \times \{|z_n| < r\}$, $r > 0$, according to the uniqueness theorem $\Delta U \equiv 0$ in polydisk ${}^{/}V \times V_n$
This means that U is harmonic in ${}^{/}V \times V_n$
From Theorem I it easily follows

Corollary I. Let the function $f({}^{/}z, z_n)$ be defined in the polydisk ${}^{/}V \times V_n = {}^{/}V \times \{|z_n| < r\} \in C^{n-1}_{{}^{/}z} \times C_{z_n}, R > 0$ and satisfies the following conditions:

1) For each fixed ${}^{/}z_0 \in {}^{/}V$ the function $f({}^{/}z, z_n)$ is holomorphic in the circle V_n
2) The function $f({}^{/}z, 0)$ is holomorphic in ${}^{/}V$ on ${}^{/}z_0$.
3) The function $\operatorname{Re} f({}^{/}z, z_n)$ is plusubharmonic in a smaller polydisk ${}^{/}V \times \{|z_n| < r\}$, $0 < r < R$.

Then $f({}^{/}z, z_n)$ extends holomorphically in ${}^{/}V \times V_n$.

Evidence. Applying the theorem I. I to the function $U = \operatorname{Re} f$, we obtain that $\operatorname{Re} f$. Is pluharmonic in ${}^{/}V \times \{|z_n| < r\}$. Consequently, there is such a function $F \in O\left({}^{/}V \times \{|z_n| < r\}\right)$ that $U = \operatorname{Re} f$. We now fix ${}^{/}z_0 \in {}^{/}V$ and consider the difference $F({}^{/}z, z_n) - f({}^{/}z, z_n)$. It is holomorphic on z_n in a circle $|z_n| < r$ and take in a purely imaginary value, as $\operatorname{Re}\left(F({}^{/}z, z_n) - f({}^{/}z, z_n)\right) = 0$.

Hence, the difference $F - f$ is in ${}^{/}V \times \{|z_n| < r\}$ of the imaginary constant dependent, it can be because of ${}^{/}z_0 \in {}^{/}V$:

$$F({}^{/}z, z_n) - f({}^{/}z, z_n) = ic({}^{/}z)$$

But when $z_n = 0$, the function $F({}^{/}z, z_n) - f({}^{/}z, z_n) = ic({}^{/}z)$ is holomorphic in ${}^{/}V$ on ${}^{/}z$. From this $ic({}^{/}z) = const$ is a purely imaginary, holomorphic function. Hence, the function $f = F - const$ is holomorphic in polydisk ${}^{/}V \times \{|z_n| < r\}$ and for each fixed ${}^{/}z$ extends holomorphically in $|z_n| < R$. Hence, according to Hartogs' Lemma (see Introduction) is holomorphic in a large polydisk ${}^{/}V \times \{|z_n| < R\}$.

The corollary is proved.

Note that an arbitrary convex function is a plusubharmonic function. So, the corollary I can also be formulated in the following form:

Corollary II. Suppose that the function $f({}^{/}z, z_n)$ is defined in polydisk ${}^{/}V \times \{|z_n| < R\} \subset C^{n-1}_{{}^{/}z} \times C_{z_n} R > 0$, and satisfies the following conditions:

1) for each fixed ${}^{/}z \in {}^{/}V$ the function $f({}^{/}z, z_n)$ is holomorphic in polydisk $|z_n| < R$

2) $f({}^{/}z, 0)$ is holomorphic in ${}^{/}V$ on ${}^{/}z$.

3) the function $\operatorname{Re} f(z, z_n)$ is convex in a smaller polydisk ${}^{/}V \times \{|z_n| < r\}, 0 < r < R$.

Then $f({}^{/}z, z_n)$ holomorphically continues ${}^{/}V \times \{|z_n| < R\}$.

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