ON THE DIOPHANTINE EQUATION $f(n) = n$

Sarita[1], Hari Kishan[2] & Megha Rani[3]
[1]Department of Mathematics, DCR University, Murthal, Sonipat (Haryana),
E-mail:sarita.malik1281@gmail.com
[2]Department of Mathematics, D.N. College, Meerut (U.P.), E-mail:harikishan10@rediffmail.com
[3]Department of Mathematics, RKGIT, Ghaziabad (U.P.), E-mail:meghar10@gmail.com

ABSTRACT

In this paper, we have discussed the Diophantine equation $f(n) = n$ where n is a positive integer with digits more than 1. Results have been obtained for number of digits of n as $2, 3, 4, 5$ and 6.

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INTRODUCTION

Brian Miceli (2015) proposed problem of week as given below:
Given a positive, three-digit integer n, define $f(n)$ to be the sum of three digits of n, their three products in pairs and the product and all three digits. Find all positive three-digit integers n such that $f(n) = n$.
In this paper, we have extended the above problem by considering the Diophantine equation $f(n) = n$ where n is a positive integer with digits more than 1. Results have been discussed for number of digits of n as $2, 3, 4, 5$ and 6.

ANALYSIS

(a) For n having 2 digits: Let the tenth digit be a and unit digit be b of n. Then

$$n = 10a + b$$

and

$$f(n) = a + b + ab$$

Therefore the given Diophantine equation reduces to

$$a + b + ab = 10a + b$$

or

$$a(b - 9) = 0. \quad \ldots(1)$$

Equation (1) gives either $a = 0$ or $b = 9$. But $a = 0$ is not possible because in this case n has only one digit. Hence $b = 9$. The tenth digit a can take any value from 1 to 9. Thus the required values of n are 19, 29, 39, 49, 59, 69, 79, 89 and 99.

(b) For n having 3 digits: Let the hundredth digit be a, the tenth digit be b and unit digit be c of n. Then

$$n = 100a + 10b + c$$

and

$$f(n) = a + b + c + ab + bc + ca + abc$$

Therefore the given Diophantine equation reduces to

$$a + b + c + ab + bc + ac + abc = 100a + 10b + c$$

or

$$(1 + c)(a + b + ab) = 100a + 10b. \quad \ldots(2)$$

Putting $c = 9$ in (2), we get

$$(a + b + ab) = 10a + b$$

This implies that

$$a(b-9)=0.$$
…(3)

Equation (3) gives either $a=0$ or $b=9$. But $a=0$ is not possible because in this case n has only two digits. Hence $b=9$. The hundredth digit a can take any value from 1 to 9. Thus the required values of *n* are 199, 299, 399, 499, 599, 699, 799, 899 and 999.

(c) For *n* having 4 digits: Let the thousandth digit be *a*, the hundredth digit be *b*, the tenth digit be *c* and unit digit be *d* of *n*. Then

$$n=1000a+100b+10c+d$$

and $f(n)=a+b+c+d+ab+bc+cd+ac+ad+bd+abc+bcd+acd+abd+abcd.$
…(4)

Therefore the given Diophantine equation reduces to

$$a+b+c+d+ab+bc+cd+ac+ad+bd+abc+bcd+acd+abd+abcd$$
$$=1000a+100b+10c+d$$

or $(1+d)(a+b+c+ab+bc+ca+abc)=1000a+100b+10c.$
…(5)

Putting $d=9$ in (5), we get

$a+b+c+ab+bc+ca+abc=100a+10b+c.$
…(6)

or $(1+c)(a+b+ab)=100a+10b.$
…(7)

Putting $c=9$ in (7), we get

$$(a+b+ab)=10a+b$$

This implies that

$$a(b-9)=0.$$
…(8)

Equation (8) gives either $a=0$ or $b=9$. But $a=0$ is not possible because in this case n has only two digits. Hence $b=9$. The thousandth digit a can take any value from 1 to 9. Thus the required values of *n* are 1999, 2999, 3999, 4999, 5999, 6999, 7999, 8999 and 9999.

(d) For *n* having 5 digits: Let the ten thousandth digit be *a*, the thousandth digit be *b*, the hundredth digit be *c*, the tenth digit be *d* and unit digit be *e* of *n*. Then

$$n=10000a+1000b+100c+10d+e$$

and $f(n)=a+b+c+d+e+ab+bc+cd+ac+ad+ae+bd+be+ce+de+abc+bcd+acd+ace+ade+bce+cde+bde+abd+abe+abcd+acde+abde+abce+bcde+abcde.$

Therefore the given Diophantine equation reduces to

$a+b+c+d+e+ab+bc+cd+ac+ad+ae+bd+be+ce+de+abc+bcd+acd+ace+ade+bce+cde+bde+abd+abe+abcd+acde+abde+abce+bcde+abcde=10000a+1000b+100c+10d+e$

or $(1+e)(a+b+c+d+ab+bc+cd+ac+ad+bd+abc+bcd+acd++abd+abcd=10000a+1000b+100c+10d.$

…(9) Putting $e=9$ in (9), we get

$(a+b+c+d+ab+bc+cd+ac+ad+bd+abc+bcd+acd++abd+abcd)=1000a+100b+10c+d.$

…(10)

or $(1+d)(a+b+c+ab+bc+ca+abc) = 1000a+100b+10c.$

…(11) Putting $d=9$ in (11), we get

$$a+b+c+ab+bc+ca+abc = 100a+10b+c$$

or $(1+c)(a+b+ab) = 100a+10b.$

…(12) Putting $c=9$ in (12), we get

$$a+b+ab = 10a+b$$

or $a(b-9)=0.$

…(13)

Equation (13) gives either $a=0$ or $b=9$. But $a=0$ is not possible because in this case n has only two digits. Hence $b=9$. The ten thousandth digit a can take any value from 1 to 9. Thus the required values of n are 19999, 29999, 39999, 49999, 59999, 69999, 79999, 89999 and 99999.

(e) For n having 6 digits: Let the digit of lac place be a, the ten thousandth digit be b, the thousandth digit be c, the hundredth digit be d, the tenth digit be e and unit digit be g of n. Then

$$n = 100000a+10000b+1000c+100d+10e+g$$

and $f(n) = a+b+c+d+e+g+ab+bc+cd+ac+ad+ae+bd+be+ce+de+ag+bg+cg+dg+eg+abc+bcd+acd+ace+ade+bce+cde+bde+abd+abe+abg+acg+adg+aeg+bcg+bdg+cdg+deg+ceg+beg+abcd+abce+abcg+acde+acdg+aceg+adeg+abde+abdg+abeg+bcde+bcdg+bceg+cdeg+abcde+bcdeg+acdeg+abdeg+abceg+abcdg+abcdeg.$

Therefore the given Diophantine equation reduces to

$a+b+c+d+e+g+ab+bc+cd+ac+ad+ae+bd+be+ce+de+ag+bg+cg+dg+eg+abc+bcd+acd+ace+ade+bce+cde+bde+abd+abe+abg+acg+adg+aeg+bcg+bdg+cdg+deg+ceg+beg+abcd+abce+abcg+acde+acdg+aceg+adeg+abde+abdg+abeg+bcde+bcdg+bceg+cdeg+abcde+bcdeg+acdeg+abdeg+abceg+abcdg+abcdeg = 100000a+10000b+1000c+100d+10e+g$

or $(1+g)(a+b+c+d+e+ab+bc+cd+ac+ad+ae+bd+be+ce+de+abc+bcd+acd+ace+ade+bce+cde+bde+abd+abe+abcd+acde+abde+abce+bcde+abcde=100000a+10000b+1000c+100d+10e.$

…(14)

Putting $g=9$ in (14), we get

$$a+b+c+d+e+ab+bc+cd+ac+ad+ae+bd+be+ce+de+abc+bcd+acd+ace+ade+bce+cde+bde+abd+abe+abcd+acde+abde+abce+bcde+abcde = 10000a+1000b+100c+10d+e$$

or $(1+e)(a+b+c+d+ab+bc+cd+ac+ad+bd+abc+bcd+acd++abd+abcd=10000a+1000b+100c+10d.$ …(15)

Putting $e=9$ in (15), we get

$(a+b+c+d+ab+bc+cd+ac+ad+bd+abc+bcd+acd++abd+abcd) = 1000a+100b+10c+d.$

…(16)

or $(1+d)(a+b+c+ab+bc+ca+abc) = 1000a+100b+10c.$

…(17) Putting $d=9$ in (17), we get

$$a+b+c+ab+bc+ca+abc = 100a+10b+c$$

or $(1+c)(a+b+ab) = 100a + 10b$.
…(18) Putting $c = 9$ in (18), we get
$a+b+ab = 10a+b$
or $a(b-9) = 0$.
…(19)
Equation (19) gives either $a = 0$ or $b = 9$. But $a = 0$ is not possible because in this case n has only two digits. Hence $b = 9$. The digit of lac place a can take any value from 1 to 9. Thus the required values of n are 199999, 299999, 399999, 499999, 599999, 699999, 799999, 899999 and 999999.

CONCLUDING REMARKS

Here the results for the Diophantine equation $f(n) = n$ have been obtained for n = $2, 3, 4, 5$ and 6. This Diophantine equation may be discussed for other values of n.

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