ON THE BEHAVIOR OF SOLUTIONS OF THE SYSTEM OF RATIONAL DIFFERENCE EQUATIONS

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ABSTRACT

In this paper, we investigated the behavaior of the positive solutions of the difference equations system

$$x_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1},\quad y_{n+1}=\frac{x_{n-1}}{x_n y_{n-1}+1},\quad z_{n+1}=\frac{z_{n-1}}{x_{n-1}y_n+y_{n-1}x_n}$$

and

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\quad y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1},\quad z_{n+1}=\frac{z_{n-1}}{x_{n-1}y_n+y_{n-1}x_n}$$

where the initial conditions are positive real numbers.

Keywords: Difference equations, difference equations systems, solutions, equilibrium point, behavior of solutions, rational difference equations, systems of rational difference equations.

INTRODUCTION

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology etc. There are many papers with related to the difference equations system for example,

In [1] A. S. Kurbanlı, C. Çinar and I. Yalcinkaya, studied the behavaior of positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\quad y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1}.$$

In [12] Cinar studied the solutions of the systems of difference equations

$$x_{n+1}=\frac{1}{y_n},\quad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

In [13] Papaschinnopoulos and Schinas studied the oscillatory behavior, the boundedness of the solutions, and the global asymptotic stability of the positive equilibrium of the system of nonlinear difference equations

$$x_{n+1}=A+\frac{y_n}{x_{n-p}},\quad y_{n+1}=A+\frac{x_n}{y_{n-q}},\ n=0,1,\ldots p,\ q.$$

In [24] Özban studied the positive solutions of the system of rational difference equations

$$x_{n+1}=1+\frac{x_n}{y_{n-k}},\ y_{n+1}=1+\frac{y_n}{x_{n-m}y_{n-m-k}}$$

In [29] Yalcinkaya studied the global asmptotic stability of the system of difference equations

$$z_{n+1}=\frac{t_n z_{n-1}+a}{t_n+z_{n-1}},\ t_{n+1}=\frac{z_n t_{n-1}+a}{z_n+t_{n-1}}$$

In [30] Yalcinkaya, Cinar and Simsek studied the global asmptotic stability of the system of difference equations

$$z_{n+1}=\frac{t_n+z_{n-1}}{t_n z_{n-1}+a},\quad t_{n+1}=\frac{z_n+t_{n-1}}{z_n t_{n-1}+a}$$

Also see reference.

In this paper, we investigated the behavaior of the positive solutions of the difference equations system

$$x_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1},\ y_{n+1}=\frac{x_{n-1}}{x_n y_{n-1}+1},\ z_{n+1}=\frac{z_{n-1}}{x_{n-1}y_n+y_{n-1}x_n} \quad (1.1)$$

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\ y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1},\ z_{n+1}=\frac{z_{n-1}}{x_{n-1}y_n+y_{n-1}x_n} \quad (1.2)$$

where the initial conditions are positive real numbers.

MAIN RESULTS

Theorem 1. Let

$$y_0=a,\ y_{-1}=b,\ x_0=c,\ x_{-1}=d,\ z_0=e,\ z_{-1}=f \text{ and } ad\neq 1,\ bc\neq 1$$

be positive real numbers and let $\{x_n,y_n,z_n\}$ be a solution of the system (1.1). Then all solutions of (1.1) are

$$x_n=\begin{cases}\dfrac{d}{(ad-1)^n}, & n-odd\\ c(bc-1)^n, & n-even\end{cases} \quad (1.3)$$

$$y_n=\begin{cases}\dfrac{b}{(bc-1)^n}, & n-odd\\ a(ad-1)^n, & n-even\end{cases} \quad (1.4)$$

$$z_n=\begin{cases}\dfrac{f}{(ad+bc)^n}, & n-odd\\ e\left[\dfrac{(bc-1)(ad-1)}{ad(bc-1)+bc(ad-1)}\right]^n, & n-even\end{cases} \quad (1.5)$$

Proof. n = 0, 1, 2 we have

$$x_1=\frac{y_{-1}}{x_0 y_{-1}+1}=\frac{d}{ad-1}$$

$$y_1=\frac{x_{-1}}{x_0y_{-1}+1}=\frac{b}{bc-1}$$

$$z_1=\frac{z_{-1}}{x_{-1}y_0+y_{-1}x_0}=\frac{f}{ad+bc}$$

$$x_2=\frac{x_0}{y_1x_0-1}=\frac{c}{\frac{b}{bc-1}c-1}=\frac{c}{\frac{bc-(bc-1)}{bc-1}}=c(bc-1)$$

$$y_2=\frac{y_0}{x_1y_0-1}=\frac{a}{\frac{d}{ad-1}a-1}=\frac{a}{\frac{ad-(ad-1)}{ad-1}}=a(ad-1)$$

$$z_2=\frac{z_0}{x_0y_1+y_0x_1}=\frac{e}{c\frac{b}{bc-1}+a\frac{d}{ad-1}}=\frac{e}{\frac{bc}{bc-1}+\frac{ad}{ad-1}}$$

$$=\frac{e}{\frac{bc(ad-1)+ad(bc-1)}{(bc-1)(ad-1)}}=\frac{e(bc-1)(ad-1)}{bc(ad-1)+ad(bc-1)}$$

For $n=$ 3, 4

$$x_3=\frac{x_1}{y_2x_1-1}=\frac{\frac{d}{ad-1}}{a(ad-1)\frac{d}{ad-1}-1}=\frac{\frac{d}{ad-1}}{(ad-1)\frac{ad}{ad-1}-1}=\frac{\frac{d}{ad-1}}{ad-1}=d(ad-1)^2$$

$$y_3=\frac{y_1}{x_2y_1-1}=\frac{\frac{b}{bc-1}}{c(bc-1)\frac{b}{bc-1}-1}=\frac{\frac{b}{bc-1}}{(bc-1)\frac{bc}{bc-1}-1}=\frac{\frac{b}{bc-1}}{bc-1}=\frac{b}{(bc-1)^2}$$

$$z_3=\frac{z_1}{x_1y_2+y_1x_2}=\frac{\frac{f}{ad+bc}}{\left(\frac{d}{ad-1}\right)(a(ad-1))+\left(\frac{b}{bc-1}\right)(c(bc-1))}$$

$$=\frac{\frac{f}{ad+bc}}{\left(\frac{ad(ad-1)}{ad-1}\right)+\left(\frac{bc(bc-1)}{bc-1}\right)}=\frac{\frac{f}{ad+bc}}{ad+bc}=\frac{f}{(ad+bc)^2}$$

for $n=k$ assume that

$$x_k=\begin{cases}\frac{d}{(ad-1)^k}, & k-\text{odd}\\ c(bc-1)^k, & k-\text{even}\end{cases}$$

$$y_k=\begin{cases}\frac{b}{(bc-1)^k}, & k-\text{odd}\\ a(ad-1)^k, & k-\text{even}\end{cases}$$

and

$$z_k = \begin{cases} \dfrac{f}{(ad+bc)^k}, & k-\text{odd} \\ e\left[\dfrac{(bc-1)(ad-1)}{ad(bc-1)+bc(ad-1)}\right]^k, & k-\text{even} \end{cases}$$

are true. Then for $n = k + 1$ will. Show that (1.3), (1.4) and (1.5) are true. From (1.1) we have

$$x_{2k+1} = \frac{x_{2k-1}}{y_{2k}x_{2k-1}-1} = \frac{\dfrac{d}{(ad-1)^k}}{a(ad-1)^k\dfrac{d}{(ad-1)^k}-1} = \frac{\dfrac{d}{(ad-1)^k}}{ad-1} = \frac{d}{(ad-1)^{k+1}}$$

$$y_{2k+1} = \frac{y_{2k-1}}{x_{2k}y_{2k-1}-1} = \frac{\dfrac{b}{(bc-1)^k}}{c(bc-1)^k\dfrac{b}{(bc-1)^k}-1} = \frac{\dfrac{b}{(bc-1)^k}}{bc-1} = \frac{b}{(bc-1)^{k+1}}$$

Also, smilarly from (1.1), we have

$$z_{2k+1} = \frac{z_{2k-1}}{x_{2k-1}y_{2k}+y_{2k-1}x_{2k}} = \frac{\dfrac{f}{(ad+bc)^k}}{\dfrac{d}{(ad-1)^k}a(ad-1)^k + \dfrac{b}{(bc-1)^k}c(bc-1)^k}$$

$$= \frac{\dfrac{f}{(ad+bc)^k}}{ad+bc} = \frac{f}{(ad+bc)^{k+1}}$$

Also, we have

$$x_{2k+2} = \frac{x_{2k}}{y_{2k+1}x_{2k}-1} = \frac{c(bc-1)^k}{\dfrac{b}{(bc-1)^{k+1}}c(bc-1)^k-1} = \frac{c(bc-1)^k}{\dfrac{bc}{(bc-1)}-1} = \frac{c(bc-1)^k}{\dfrac{1}{(bc-1)}} = c(bc-1)^{k+1}$$

$$y_{2k+2} = \frac{y_{2k}}{x_{2k+1}y_{2k}-1} = \frac{a(ad-1)^k}{\dfrac{d}{(ad-1)^{k+1}}a(ad-1)^k-1} = \frac{a(ad-1)^k}{\dfrac{ad}{(ad-1)}-1} = \frac{a(ad-1)^k}{\dfrac{1}{(ad-1)}} = a(ad-1)^{k+1}$$

and

$$z_{2k+2} = \frac{z_{2k}}{x_{2k}y_{2k+1}+y_{2k}x_{2k+1}} = \frac{e\left(\dfrac{(bc-1)^k(ad-1)^k}{[ad(bc-1)+bc(ad-1)]^k}\right)}{c(bc-1)^k\dfrac{b}{(bc-1)^{k+1}} + a(ad-1)^k\dfrac{d}{(ad-1)^{k+1}}}$$

$$=\frac{e\left(\dfrac{(bc-1)^k(ad-1)^k}{\left[ad(bc-1)+bc(ad-1)\right]^k}\right)}{\dfrac{bc}{(bc-1)}+\dfrac{ad}{(ad-1)}}=\frac{e\left(\dfrac{(bc-1)^k(ad-1)^k}{\left[ad(bc-1)+bc(ad-1)\right]^k}\right)}{\dfrac{bc(ad-1)+ad(bc-1)}{(bc-1)(ad-1)}}=\frac{e\left(\dfrac{(bc-1)^k(ad-1)^k}{\left[ad(bc-1)+bc(ad-1)\right]^k}\right)}{\dfrac{bc(ad-1)+ad(bc-1)}{(bc-1)(ad-1)}}$$

$$=e\left(\frac{(bc-1)^k(ad-1)^k}{\left[ad(bc-1)+bc(ad-1)\right]^k}\right)\left(\frac{(bc-1)(ad-1)}{bc(ad-1)+ad(bc-1)}\right)$$

$$=e\left(\frac{(bc-1)^{k+1}(ad-1)^{k+1}}{\left[ad(bc-1)+bc(ad-1)\right]^{k+1}}\right)=e\left[\frac{(bc-1)(ad-1)}{ad(bc-1)+bc(ad-1)}\right]^{k+1}.$$

Corollary 1. Let $y_0 = a, y_{-1} = b, x_0 = c, x_{-1} = b$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$ be positive real numbers and $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $0 < a,b,c,d < 1$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \begin{cases} -\infty, & n-\text{odd} \\ +\infty, & n-\text{even} \end{cases}$$

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$$

and

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} \infty \quad , & ad+bc \in (0,1) \\ 0 \quad , & ad+bc \in (1,2) \end{cases}$$

$$\lim_{n\to\infty} z_{2n} = \begin{cases} \infty, & -2 < \dfrac{1}{(ad-1)} + \dfrac{1}{(bc-1)} < -1 \\ 0, & -1 < \dfrac{1}{(ad-1)} + \dfrac{1}{(bc-1)} < \infty \end{cases}.$$

Proof: From $a,b,c,d,e,f \in \mathbb{R}^+$, $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$, $ad \neq -bc$ and $0 < a,d < 1 \Rightarrow 0 < ad < 1 \Rightarrow -1 < ad - 1 < 0.$
Hence, we obtain

$$-1 < ad-1 < 0 \Rightarrow -1 < \frac{1}{ad-1} < \infty,$$

$$0 < b,c < 1 \Rightarrow 0 < bc < 1 \Rightarrow -1 < bc-1 < 0,$$

$$-1 < bc-1 < 0 \Rightarrow -1 < \frac{1}{bc-1} < \infty,$$

$$-1 < \frac{1}{ad-1} < \infty \, ve \, -1 < \frac{1}{bc-1} < \infty \Rightarrow -2 < \frac{1}{ad-1} + \frac{1}{bc-1} < \infty$$

$$0 < 2 + \frac{1}{ad-1} + \frac{1}{bc-1} < \infty \Rightarrow 0 < \frac{1}{2 + \dfrac{1}{ad-1} + \dfrac{1}{bc-1}} < \infty$$

$$0 < a,b,c,d < 1 \Rightarrow 0 < ad < 1 \; ve \; 0 < bc < 1 \Rightarrow 0 < ad + bc < 2$$

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = \begin{cases} -\infty, & n-odd \\ +\infty, & n-\text{even} \end{cases}$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(bc-1)^n} = b\lim_{n\to\infty} \frac{1}{(bc-1)^n} = \begin{cases} -\infty, & n-odd \\ +\infty, & n-even \end{cases}$$

and

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} \lim_{n\to\infty} \frac{f}{(ad+bc)^n} = f\lim_{n\to\infty} \frac{1}{(ad+bc)^n} = f.\infty = \infty, & ad+bc \in (0,1) \\ \lim_{n\to\infty} \frac{f}{(ad+bc)^n} = f\lim_{n\to\infty} \frac{1}{(ad+bc)^n} = f.0 = 0, & ad+bc \in (1,2) \end{cases}$$

Smilarly, we have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(bc-1)^n = c\lim_{n\to\infty} (bc-1)^n = c.0 = 0$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a\lim_{n\to\infty} (ad-1)^n = a.0 = 0$$

and

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} e\left[\frac{(bc-1)(ad-1)}{ad(bc-1)+bc(ad-1)}\right]^n = e\lim_{n\to\infty}\left[\frac{\frac{(bc-1)(ad-1)}{(bc-1)(ad-1)}}{\frac{ad(bc-1)+bc(ad-1)}{(bc-1)(ad-1)}}\right]^n$$

$$= e\lim_{n\to\infty}\left[\frac{1}{\frac{ad}{(ad-1)}+\frac{bc}{(bc-1)}}\right]^n = e\lim_{n\to\infty}\left[\frac{1}{\frac{ad-1+1}{(ad-1)}+\frac{bc-1+1}{(bc-1)}}\right]^n$$

$$= e\lim_{n\to\infty}\left[\frac{1}{1+\frac{1}{(ad-1)}+1+\frac{1}{(bc-1)}}\right]^n = e\lim_{n\to\infty}\left[\frac{1}{2+\frac{1}{(ad-1)}+\frac{1}{(bc-1)}}\right]^n$$

$$= \begin{cases} e.\infty, & -2 < \frac{1}{(ad-1)}+\frac{1}{(bc-1)} < -1 \\ e.0, & -1 < \frac{1}{(ad-1)}+\frac{1}{(bc-1)} < \infty \end{cases} = \begin{cases} \infty, & -2 < \frac{1}{(ad-1)}+\frac{1}{(bc-1)} < -1 \\ 0, & -1 < \frac{1}{(ad-1)}+\frac{1}{(bc-1)} < \infty \end{cases}.$$

Corollary 2. Let $y_0 = a, y_{-1} = b, x_0 = c, x_{-1} = b$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$ be positive real numbers and $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $1 < ad, bc < 2$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \begin{cases} -\infty, & d < 0 \\ +\infty, & d > 0 \end{cases}$$

$$\lim_{n\to\infty} y_{2n-1} = \begin{cases} -\infty, & d < 0 \\ +\infty, & d > 0 \end{cases}$$

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0$$

and

$$\lim_{n\to\infty} z_{2n-1} = 0, \ \lim_{n\to\infty} z_{2n} = 0.$$

Proof: From $a,b,c,d,e,f \in \mathbb{R}^+$, $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$, $ad \neq -bc$ and $1 < ad < 2 \Rightarrow 0 < ad - 1 < 1$

Hence, we obtain

$$\lim_{n\to\infty} (ad-1)^n = 0$$

$$1 < bc < 2 \Rightarrow 0 < bc - 1 < 1$$

Hence, we obtain

$$\lim_{n\to\infty} (bc-1)^n = 0.$$

Then

$$1 < ad < 2 \ ve \ 1 < bc < 2 \Rightarrow 2 < ad + bc < 4 .$$

Hence, we obtain

$$\lim_{n\to\infty} (ad+bc)^n = \infty$$

and

$$0 < ad - 1 < 1 \Rightarrow 1 < \frac{1}{ad-1} < \infty ,$$

$$0 < bc - 1 < 1 \Rightarrow 1 < \frac{1}{bc-1} < \infty ,$$

$$1 < \frac{1}{ad-1} < \infty \, ve \; 1 < \frac{1}{bc-1} < \infty \Rightarrow 2 < \frac{1}{ad-1} + \frac{1}{bc-1} < \infty ,$$

$$\Rightarrow 4 < 2 + \frac{1}{ad-1} + \frac{1}{bc-1} < \infty \Rightarrow 0 < \frac{1}{2 + \frac{1}{ad-1} + \frac{1}{bc-1}} < \frac{1}{4}$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{2 + \frac{1}{ad-1} + \frac{1}{bc-1}} \right)^n = 0$$

According to this

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.\infty = \begin{cases} -\infty, & d < 0 \\ +\infty, & d > 0 \end{cases}$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(bc-1)^n} = b \lim_{n\to\infty} \frac{1}{(bc-1)^n} = b.\infty = \begin{cases} -\infty, & b < 0 \\ +\infty, & b > 0 \end{cases}$$

$$\lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} \frac{f}{(ad+bc)^n} = f \lim_{n\to\infty} \frac{1}{(ad+bc)^n} = f.0 = 0$$

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(bc-1)^n = c \lim_{n\to\infty} (bc-1)^n = c.0 = 0$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a \lim_{n\to\infty} (ad-1)^n = a.0 = 0$$

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} e \left[\frac{1}{2 + \frac{1}{(ad-1)} + \frac{1}{(bc-1)}} \right]^n = e \lim_{n\to\infty} \left[\frac{1}{2 + \frac{1}{(ad-1)} + \frac{1}{(bc-1)}} \right]^n = e.0 = 0$$

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Corollary 3. Let $y_0 = a, y_{-1} = b, x_0 = c, x_{-1} = b$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$ be positive real numbers and $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $-\infty < ad, bc < -1$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} z_{2n} = 0$$
$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = \infty .$$

Proof: From $a,b,c,d,e,f \in \mathbb{R}^+$, $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$, $ad \neq -bc$, $-\infty < ad < -1 \Rightarrow -\infty < ad - 1 < -2$ and $1 < ad < 2 \Rightarrow 0 < ad - 1 < 1$
Hence, we obtain

$$\lim_{n\to\infty} (ad-1)^n = \begin{cases} -\infty, & n - \text{odd} \\ +\infty, & n - \text{even} \end{cases}$$

and

$$-\infty < bc < -1 \Rightarrow -\infty < bc - 1 < -2 \Rightarrow \lim_{n\to\infty} (bc-1)^n = \begin{cases} -\infty, & n - \text{odd} \\ +\infty, & n - \text{even} \end{cases},$$
$$-\infty < ad < -1 \; ve \; -\infty < bc < -1 \Rightarrow -\infty < ad + bc < -2$$
$$\Rightarrow \lim_{n\to\infty} (ad+bc)^n = \begin{cases} -\infty, & n - odd \\ +\infty, & n - even \end{cases}$$

From this

$$-\infty < ad - 1 < -2 \Rightarrow -\frac{1}{2} < \frac{1}{ad-1} < 0,$$
$$-\infty < bc - 1 < -2 \Rightarrow -\frac{1}{2} < \frac{1}{bc-1} < 0,$$
$$-\frac{1}{2} < \frac{1}{ad-1} < 0 \;\; and \;\; -\frac{1}{2} < \frac{1}{bc-1} < 0 \Rightarrow -1 < \frac{1}{ad-1} + \frac{1}{bc-1} < 0$$
$$-1 < \frac{1}{ad-1} + \frac{1}{bc-1} < 0 \Rightarrow 1 < 2 + \frac{1}{ad-1} + \frac{1}{bc-1} < 2$$
$$\Rightarrow \frac{1}{2} < \frac{1}{2 + \frac{1}{ad-1} + \frac{1}{bc-1}} < 1 \Rightarrow \lim_{n\to\infty} \left(\frac{1}{2 + \frac{1}{ad-1} + \frac{1}{bc-1}} \right)^n = 0 .$$

According to this

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.0 = 0$$
$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(bc-1)^n} = b \lim_{n\to\infty} \frac{1}{(bc-1)^n} = b.0 = 0$$

and

$$\lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} \frac{f}{(ad+bc)^n} = f \lim_{n\to\infty} \frac{1}{(ad+bc)^n} = f.0 = 0$$

Similarly

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(bc-1)^n = c \lim_{n\to\infty} (bc-1)^n$$

$$= c\begin{cases} -\infty, & n-\text{odd} \\ +\infty, & n-\text{even} \end{cases} = \begin{cases} -\infty \ , & n-odd \ and \ c>0 \\ -\infty \ , & n-even \ and \ c<0 \\ +\infty \ , & n-odd \ and \ c<0 \\ +\infty \ , & n-even \ and \ c>0 \end{cases}$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a\left(ad-1\right)^n = a\lim_{n\to\infty}\left(ad-1\right)^n = a.0 = 0$$

and

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} e\left[\frac{1}{2+\frac{1}{(ad-1)}+\frac{1}{(bc-1)}}\right]^n$$

$$= e\lim_{n\to\infty}\left[\frac{1}{2+\frac{1}{(ad-1)}+\frac{1}{(bc-1)}}\right]^n = e.0 = 0$$

Corollary 4. Let $y_0 = a, y_{-1} = b, x_0 = c, x_{-1} = b$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$ be positive real numbers and $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $2 < ad, bc < \infty$ then we have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} z_{2n} = 0$$

$$\lim_{n\to\infty} x_{2n} = \begin{cases} +\infty, & c>0 \\ -\infty, & c<0 \end{cases}$$

and

$$\lim_{n\to\infty} y_{2n} = \begin{cases} +\infty, & a>0 \\ -\infty, & a<0 \end{cases}.$$

Proof: From $a,b,c,d,e,f \in \mathbb{R}^{+}$, $y_0 = a$, $y_{-1} = b$, $x_0 = c$, $x_{-1} = d$, $z_0 = e$, $z_{-1} = f$, $ad \neq 1$, $bc \neq 1$, $ad \neq -bc$. If $2 < ad < \infty$, $2 < bc < \infty$, $2 < ad < \infty$ then we have

$$2 < ad < \infty \Rightarrow 1 < ad-1 < \infty \Rightarrow \lim_{n\to\infty}\left(ad-1\right)^n = \infty,$$

$$2 < bc < \infty \Rightarrow 1 < bc-1 < \infty \Rightarrow \lim_{n\to\infty}\left(bc-1\right)^n = \infty,$$

$$2 < ad < \infty \ and \ 2 < bc < \infty \Rightarrow 4 < ad+bc < \infty \Rightarrow \lim_{n\to\infty}\left(ad+bc\right)^n = \infty$$

$$1 < ad-1 < \infty \Rightarrow 0 < \frac{1}{ad-1} < 1 \text{ and } 1 < bc-1 < \infty \Rightarrow 0 < \frac{1}{bc-1} < 1,$$

$$0 < \frac{1}{ad-1} < 1 \ and \ 0 < \frac{1}{bc-1} < 1 \Rightarrow 0 < \frac{1}{ad-1} + \frac{1}{bc-1} < 2$$

$$\Rightarrow 2 < 2 + \frac{1}{ad-1} + \frac{1}{bc-1} < 4 \Rightarrow \frac{1}{4} < \frac{1}{2+\frac{1}{ad-1}+\frac{1}{bc-1}} < \frac{1}{2}$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{2+\frac{1}{ad-1}+\frac{1}{bc-1}}\right)^n=0.$$

According to this

$$\lim_{n\to\infty} x_{2n-1}=\lim_{n\to\infty}\frac{d}{(ad-1)^n}=d\lim_{n\to\infty}\frac{1}{(ad-1)^n}=d.0=0$$

$$\lim_{n\to\infty} y_{2n-1}=\lim_{n\to\infty}\frac{b}{(bc-1)^n}=b\lim_{n\to\infty}\frac{1}{(bc-1)^n}=b.0=0$$

$$\lim_{n\to\infty} z_{2n-1}=\lim_{n\to\infty}\frac{f}{(ad+bc)^n}=f\lim_{n\to\infty}\frac{1}{(ad+bc)^n}=f.0=0$$

Smilarly,

$$\lim_{n\to\infty} x_{2n}=\lim_{n\to\infty} c(bc-1)^n=c\lim_{n\to\infty}(bc-1)^n=c.\infty=\begin{cases}+\infty, & c>0\\ -\infty, & c<0\end{cases}$$

$$\lim_{n\to\infty} y_{2n}=\lim_{n\to\infty} a(ad-1)^n=a\lim_{n\to\infty}(ad-1)^n=a.\infty=\begin{cases}+\infty, & a>0\\ -\infty, & a<0\end{cases}$$

and

$$\lim_{n\to\infty} z_{2n}=\lim_{n\to\infty} e\left[\frac{1}{2+\frac{1}{(ad-1)}+\frac{1}{(bc-1)}}\right]^n=e\lim_{n\to\infty}\left[\frac{1}{2+\frac{1}{(ad-1)}+\frac{1}{(bc-1)}}\right]^n=e.0=0$$

Corollary 5. Let $y_0=a, y_{-1}=b, x_0=c, x_{-1}=b$, $z_0=e$, $z_{-1}=f$, $ad\neq 1$, $bc\neq 1$ be positive real numbers and $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $-1<a,b,c,d,e,f<0$ then we have

$$\lim_{n\to\infty} x_{2n-1}=\begin{cases}-\infty, & d<0\\ +\infty, & d>0\end{cases}$$

$$\lim_{n\to\infty} y_{2n-1}=\begin{cases}-\infty, & d<0\\ +\infty, & d>0\end{cases}$$

$$\lim_{n\to\infty} x_{2n}=\lim_{n\to\infty} y_{2n}=0$$

and

$$\lim_{n\to\infty} z_{2n-1}=0$$

$$\lim_{n\to\infty} z_{2n}=0$$

Proof: From $a,b,c,d,e,f\in$ □, $y_0=a$, $y_{-1}=b$, $x_0=c$, $x_{-1}=d$, $z_0=e$, $z_{-1}=f$, $ad\neq 1$, $bc\neq 1$, $ad\neq -bc$, $-1<a,b,c,d,e,f<0$

Hence, we obtain

$$-1<a,d<0\Rightarrow 0<ad<1\Rightarrow -1<ad-1<0,$$

$$-1<ad-1<0\Rightarrow -1<\frac{1}{ad-1}<\infty,$$

$$-1<b,c<0\Rightarrow 0<bc<1\Rightarrow -1<bc-1<0,$$

$$-1 < bc-1 < 0 \Rightarrow -1 < \frac{1}{bc-1} < \infty,$$

$$-1 < \frac{1}{ad-1} < \infty \ and \ -1 < \frac{1}{bc-1} < \infty \Rightarrow -2 < \frac{1}{ad-1} + \frac{1}{bc-1} < \infty$$

$$0 < 2 + \frac{1}{ad-1} + \frac{1}{bc-1} < \infty \Rightarrow 0 < \frac{1}{2 + \frac{1}{ad-1} + \frac{1}{bc-1}} < \infty$$

$$-1 < a,b,c,d < 0 \Rightarrow 0 < ad < 1 \ and \ 0 < bc < 1 \Rightarrow 0 < ad + bc < 2$$

According to this

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = \begin{cases} +\infty, & n-\text{odd} \\ -\infty, & n-\text{even} \end{cases}$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(bc-1)^n} = b \lim_{n\to\infty} \frac{1}{(bc-1)^n} = \begin{cases} +\infty, & n-\text{odd} \\ -\infty, & n-\text{even} \end{cases}$$

and

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} \lim_{n\to\infty} \frac{f}{(ad+bc)^n} = f.\infty = -\infty, & ad+bc \in (0,1) \\ \lim_{n\to\infty} \frac{f}{(ad+bc)^n} = f.0 = 0, & ad+bc \in (1,2) \end{cases}$$

Similarly

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(bc-1)^n = c \lim_{n\to\infty} (bc-1)^n = c.0 = 0$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a \lim_{n\to\infty} (ad-1)^n = a.0 = 0$$

and

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} e\left[\frac{(bc-1)(ad-1)}{ad(bc-1)+bc(ad-1)}\right]^n = e \lim_{n\to\infty}\left[\frac{\frac{(bc-1)(ad-1)}{(bc-1)(ad-1)}}{\frac{ad(bc-1)+bc(ad-1)}{(bc-1)(ad-1)}}\right]^n$$

$$= e \lim_{n\to\infty}\left[\frac{1}{\frac{ad}{(ad-1)} + \frac{bc}{(bc-1)}}\right]^n = e \lim_{n\to\infty}\left[\frac{1}{\frac{ad-1+1}{(ad-1)} + \frac{bc-1+1}{(bc-1)}}\right]^n$$

$$= e \lim_{n\to\infty}\left[\frac{1}{1 + \frac{1}{(ad-1)} + 1 + \frac{1}{(bc-1)}}\right]^n = e \lim_{n\to\infty}\left[\frac{1}{2 + \frac{1}{(ad-1)} + \frac{1}{(bc-1)}}\right]^n$$

$$=\begin{cases} e.\infty, & 0<\dfrac{1}{2+\dfrac{1}{(ad-1)}+\dfrac{1}{(bc-1)}}<1 \\ e.0, & 1<\dfrac{1}{2+\dfrac{1}{(ad-1)}+\dfrac{1}{(bc-1)}}<\infty \end{cases}$$

$$=\begin{cases} -\infty, & 0<\dfrac{1}{2+\dfrac{1}{(ad-1)}+\dfrac{1}{(bc-1)}}<1 \\ 0, & 1<\dfrac{1}{2+\dfrac{1}{(ad-1)}+\dfrac{1}{(bc-1)}}<\infty \end{cases}.$$

Theorem 2. Let $y_0=a,\ y_{-1}=b,\ x_0=c,\ x_{-1}=d,\ z_0=e,\ z_{-1}=f$ and $ad\neq 1,\ bc\neq 1$ be positive real numbers and let $\{x_n,y_n,z_n\}$ be a solution of the system (1.2). Then all solutions of (1.2) are

$$(1.6)\qquad x_n=\begin{cases} \dfrac{d\prod_{i=1}^{n}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{n}\left[(2i-1)ad+1\right]}, & n-\text{odd} \\ \dfrac{c\prod_{i=1}^{n}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{n}\left[(2i)bc+1\right]}, & n-\text{even} \end{cases}$$

$$(1.7)\qquad y_n=\begin{cases} \dfrac{b\prod_{i=1}^{n}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{n}\left[(2i-1)bc+1\right]}, & n-\text{odd} \\ \dfrac{a\prod_{i=1}^{n}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{n}\left[(2i)ad+1\right]}, & n-\text{even} \end{cases}$$

$$(1.8)\qquad z_n=\begin{cases} \dfrac{f\prod_{i=1}^{n}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]}{\prod_{i=1}^{n}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}}, & n-\text{odd} \\ \dfrac{e\prod_{i=1}^{n}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]}{\prod_{i=1}^{n}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}}, & n-\text{even} \end{cases}$$

Proof. $n = 0, 1, 2$ we have

$$x_1=\frac{x_{-1}}{y_0x_{-1}+1}=\frac{d}{ad+1}$$

$$y_1 = \frac{y_{-1}}{x_0 y_{-1}+1} = \frac{b}{bc+1}$$

$$z_1 = \frac{z_{-1}}{x_{-1}y_0 + y_{-1}x_0} = \frac{f}{ad+bc}$$

$$x_2 = \frac{x_0}{y_1 x_0 + 1} = \frac{c}{\frac{b}{bc+1}c+1} = \frac{c(bc+1)}{2bc+1}$$

$$y_2 = \frac{y_0}{x_1 y_0 + 1} = \frac{a}{\frac{d}{ad+1}a+1} = \frac{a(ad+1)}{2ad+1}$$

$$z_2 = \frac{z_0}{x_0 y_1 + y_0 x_1} = \frac{e}{c\frac{b}{bc+1}+a\frac{d}{ad+1}} = \frac{e}{\frac{bc}{bc+1}+\frac{ad}{ad+1}}$$

$$= \frac{e}{\frac{bc(ad+1)+ad(bc+1)}{(bc+1)(ad+1)}} = \frac{e(bc+1)(ad+1)}{bc(ad+1)+ad(bc+1)}$$

and

$$x_3 = \frac{x_1}{y_2 x_1 + 1} = \frac{\frac{d}{ad+1}}{\frac{a(ad+1)}{2ad+1}\frac{d}{ad+1}+1} = \frac{\frac{d}{ad+1}}{\frac{ad}{2ad+1}+1} = \frac{\frac{d}{ad+1}}{\frac{3ad+1}{2ad+1}} = \frac{d(2ad+1)}{(ad+1)(3ad+1)}$$

$$y_3 = \frac{y_1}{x_2 y_1 + 1} = \frac{\frac{b}{bc+1}}{\frac{c(bc+1)}{2bc+1}\frac{b}{bc+1}+1} = \frac{\frac{b}{bc+1}}{\frac{bc}{2bc+1}+1} = \frac{\frac{b}{bc+1}}{\frac{3bc+1}{2bc+1}} = \frac{b(2bc+1)}{(bc+1)(3bc+1)}$$

$$z_3 = \frac{z_1}{x_1 y_2 + y_1 x_2} = \frac{\frac{f}{ad+bc}}{\frac{d}{ad+1}\frac{a(ad+1)}{2ad+1}+\frac{b}{bc+1}\frac{c(bc+1)}{2bc+1}} = \frac{\frac{f}{ad+bc}}{\frac{ad}{2ad+1}+\frac{bc}{2bc+1}}$$

$$= \frac{\frac{f}{ad+bc}}{\frac{ad(2bc+1)+bc(2ad+1)}{(2ad+1)(2bc+1)}} = \frac{f(2ad+1)(2bc+1)}{(ad+bc)(ad(2bc+1)+bc(2ad+1))}$$

for $n = k$ assume that

$$x_{2k-1} = \frac{d\prod_{i=1}^{k}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}$$

$$x_{2k} = \frac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}$$

$$y_{2k-1}=\frac{b\prod_{i=1}^{k}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}$$

$$y_{2k}=\frac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}$$

and

$$z_{2k-1}=\frac{f\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}}$$

$$z_{2k}=\frac{e\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}}$$

are true. Then for $n = k + 1$ will. Show that (1.6), (1.7) and (1.8) are true. From (1.2) we have

$$x_{2k+1}=\frac{x_{2k-1}}{y_{2k}x_{2k-1}+1}=\frac{\dfrac{d\prod_{i=1}^{k}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}}{\left(\dfrac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}\right)\left(\dfrac{d\prod_{i=1}^{k}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}\right)+1}$$

$$=\frac{\dfrac{d\prod_{i=1}^{k}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}}{\left(\dfrac{ad}{(2k)ad+1}\right)+1}=\frac{\dfrac{d\prod_{i=1}^{k}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}}{\left(\dfrac{ad+(2k)ad+1}{(2k)ad+1}\right)}$$

$$=\frac{d\prod_{i=1}^{k}\left[(2i-2)ad+1\right]\left((2k)ad+1\right)}{\prod_{i=1}^{k}\left[(2i-1)ad+1\right]\left((2k+1)ad+1\right)}=\frac{d\prod_{i=1}^{k+1}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)ad+1\right]}$$

$$y_{2k+1}=\frac{y_{2k-1}}{x_{2k}y_{2k-1}+1}=\frac{\dfrac{b\prod_{i=1}^{k}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}}{\left(\dfrac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}\right)\left(\dfrac{b\prod_{i=1}^{k}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}\right)+1}$$

$$=\frac{\dfrac{b\prod_{i=1}^{k}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}}{\dfrac{bc}{\left[(2k)bc+1\right]}+1}=\frac{\dfrac{b\prod_{i=1}^{k}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}}{\dfrac{bc+(2k)bc+1}{(2k)bc+1}}$$

$$=\frac{b\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left((2k)bc+1\right)}{\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left((2k+1)bc+1\right)}=\frac{b\prod_{i=1}^{k+1}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)bc+1\right]}$$

$$x_{2k+2}=\frac{x_{2k}}{y_{2k+1}x_{2k}+1}=\frac{\dfrac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}}{\left(\dfrac{b\prod_{i=1}^{k+1}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)bc+1\right]}\right)\left(\dfrac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}\right)+1}$$

$$=\frac{\dfrac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}}{\dfrac{bc}{\left[(2(k+1)-1)bc+1\right]}+1}=\frac{\dfrac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}}{\dfrac{bc+(2k+1)bc+1}{\left[(2k+1)bc+1\right]}}$$

$$=\frac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2k+1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]\left[(2k+2)bc+1\right]}=\frac{c\prod_{i=1}^{k+1}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k+1}\left[(2i)bc+1\right]}$$

$$y_{2k+2}=\frac{y_{2k}}{x_{2k+1}y_{2k}+1}=\frac{\dfrac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}}{\left(\dfrac{d\prod_{i=1}^{k+1}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)ad+1\right]}\right)\left(\dfrac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}\right)+1}$$

$$=\frac{\dfrac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}}{\left(\dfrac{ad}{\left[(2k+1)ad+1\right]}\right)+1}=\frac{\dfrac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}}{\dfrac{ad+(2k+1)ad+1}{(2k+1)ad+1}}$$

$$=\frac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]\left[(2k+1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]\left[(2k+2)ad+1\right]}=\frac{a\prod_{i=1}^{k+1}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k+1}\left[(2i)ad+1\right]}$$

and

$$z_{2k+1}=\frac{z_{2k-1}}{x_{2k-1}y_{2k}+y_{2k-1}x_{2k}}$$

$$=\frac{\dfrac{f\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}}}{\left(\dfrac{d\prod_{i=1}^{k}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}\right)\left(\dfrac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}\right)+\left(\dfrac{b\prod_{i=1}^{k}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}\right)\left(\dfrac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}\right)}$$

$$=\frac{\dfrac{f\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}}}{\left(\dfrac{ad\prod_{i=1}^{k}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}\right)+\left(\dfrac{bc\prod_{i=1}^{k}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}\right)}=\frac{\dfrac{f\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}}}{\left(\dfrac{ad}{\left[(2k)ad+1\right]}\right)+\left(\dfrac{bc}{\left[(2k)bc+1\right]}\right)}$$

$$=\frac{\dfrac{f\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}}}{\dfrac{ad\left[(2k)bc+1\right]+bc\left[(2k)ad+1\right]}{\left[(2k)ad+1\right]\left[(2k)bc+1\right]}}$$

$$=\frac{f\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]\left[(2k)ad+1\right]\left[(2k)bc+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}\left\{ad\left[(2k)bc+1\right]+bc\left[(2k)ad+1\right]\right\}}$$

$$=\frac{f\prod_{i=1}^{k}\left[(2i-2)bc+1\right]\left[(2k)bc+1\right]\left[(2i-2)ad+1\right]\left[(2k)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}\left\{bc\left[(2k)ad+1\right]+ad\left[(2k)bc+1\right]\right\}}$$

$$=\frac{f\prod_{i=1}^{k+1}\left[(2i-2)bc+1\right]\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k+1}\left\{bc\left[(2i-2)ad+1\right]+ad\left[(2i-2)bc+1\right]\right\}}$$

$$z_{2k+2}=\frac{z_{2k}}{x_{2k}y_{2k+1}+y_{2k}x_{2k+1}}$$

$$=\frac{\dfrac{e\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}}}{\left(\dfrac{c\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k}\left[(2i)bc+1\right]}\right)\left(\dfrac{b\prod_{i=1}^{k+1}\left[(2i-2)bc+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)bc+1\right]}\right)+\left(\dfrac{a\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left[(2i)ad+1\right]}\right)\left(\dfrac{d\prod_{i=1}^{k+1}\left[(2i-2)ad+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)ad+1\right]}\right)}$$

$$= \frac{\dfrac{e\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}}}{\left(\dfrac{bc\prod_{i=1}^{k}\left[(2i-1)bc+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)bc+1\right]}\right)+\left(\dfrac{ad\prod_{i=1}^{k}\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k+1}\left[(2i-1)ad+1\right]}\right)} = \frac{\dfrac{e\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}}}{\left(\dfrac{bc}{\left[(2k+1)bc+1\right]}\right)+\left(\dfrac{ad}{\left[(2k+1)ad+1\right]}\right)}$$

$$= \frac{\dfrac{e\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}}}{\dfrac{bc\left[(2k+1)ad+1\right]+ad\left[(2k+1)bc+1\right]}{\left[(2k+1)bc+1\right]\left[(2k+1)ad+1\right]}}$$

$$= \frac{e\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]\left[(2k+1)bc+1\right]\left[(2k+1)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}\left\{bc\left[(2k+1)ad+1\right]+ad\left[(2k+1)bc+1\right]\right\}}$$

$$= \frac{e\prod_{i=1}^{k}\left[(2i-1)bc+1\right]\left[(2k+1)bc+1\right]\left[(2i-1)ad+1\right]\left[(2k+1)ad+1\right]}{\prod_{i=1}^{k}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}\left\{bc\left[(2k+1)ad+1\right]+ad\left[(2k+1)bc+1\right]\right\}}$$

$$= \frac{e\prod_{i=1}^{k+1}\left[(2i-1)bc+1\right]\left[(2i-1)ad+1\right]}{\prod_{i=1}^{k+1}\left\{bc\left[(2i-1)ad+1\right]+ad\left[(2i-1)bc+1\right]\right\}}.$$

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