ON AN INVESTIGATION OF THE MATRIX OF THE SECOND PARTIAL DERIVATIVE IN ONE ECONOMIC DYNAMICS MODEL

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ABSTRACT

The article deals with the model of economic dynamics of the Leontief type with *n* sector. We study the change rate of the sectors states depending on changes in the price vector. This equilibrium vector of the model is a solution of some functional equation. We study the properties of the matrix of the second partial derivatives of the left hand side of this equation. An estimate is derived for the norm of the growth rate of the industries in particular. For this purpose the properties of Metzler matrix is used.

Keywords: Economic dynamics, equilibrium state, matrix.

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INTRODUCTION

The article [1] considers a model of economic dynamics of the Leontiev type with *n* sectors, i.e. it is assumed that each sector produces a single product, and vice versa, each product is made by only one sector. Production activities of the sectors are described by the production functions $\mathrm{F_i}\ (i=\overline{1,n})$ for which $F_i(0)=0\quad (i=\overline{1,n})$. It is also assumed, that these functions are twice continuously differentiable and strictly superlinear. This means that the functions $\mathrm{F_i}$ are concave, polynomially uniform and satisfy the inequality $\mathrm{F^i}(x+y)>F^i(x)+F^i(y)$ if x is disproportionate to y. State of the i-th sector is given by n-dimensional vector $x^i=(x_1^i,...,x_n^i),\ \ k$-th element of which indicates the amount of k-th product at the disposal of this sector. As a result of production activity the vector x^i partially turns to the vector $B^i x^i=(v^{i_i}x_1^i,...,v^{i_n}x_n^i)$. Diagonal matrix B_i with diagonal vector $(v^{i_i},...,v^{i_n})$ $0\le v^{ik}\le 1$ is called a preservation matrix.

Thus, expending at the beginning of the time period vector x^i to its end the sector will have the vector $(y_1^i,...,y_n^i)$, where $y_k^i=v^{ik}x_k^i\quad (i\ne k),\ y_i^i=v^{ij}x_i^i+F^i(x^i)$. State of the whole modeled system is presented by the vector $X=\left(x^1,...,x^n\right)\in\left(R_+^n\right)^n$, where $x^i\in R_+^n$ is a state of the $i-$th sector.

In study of the given model the model Ω is used with fixed budget [2]

$$\Omega=\left(\{y\},U,\Lambda\right),$$

where y is some element of the cone $R_+^n, U=(U^1,...,U^n)$, where U^i is an utility function defined by the relation

$$U^i\left(\bar{f},x^i\right)=\left[\bar{f},B^i x^i\right]+\bar{f}^i F^i(x^i),\ (i=\overline{1,n}).$$

Note that the function $U^i\ (i=\overline{1,n})$ presents the cost of all funds belonging to the corresponding sector with process $\bar{f}=\left(\bar{f}^1,...,\bar{f}^n\right)$.

Λ in the model Ω stands for the vector $\Lambda=(\lambda_1,...,\lambda_n)$, the coordinate λ_i of which is a given budget of the $i-$th sector.

These of the vectors $\left(p,\bar{x}^1,...,\bar{x}^n\right)$ forms the equilibrium state in the model Ω, if the vectors $\bar{x}^i$ are solutions of the problem

$$U^i(\bar{f},x^i)\to\max \qquad (i=\overline{1,n})$$

within the conditions $[p,x^i]=\lambda_i,\quad x^i\geq 0,\ \sum_{i=1}^{n}\bar{x}^i=y,\ \ p\geq 0$.

Let $\left(p,\bar{x}^1,...,\bar{x}^n\right)$ be equilibrium state of the model Ω.

Further the convex programming problem

$$\sum_{i=1}^{n}\tilde{\lambda}_i\ln U^i(f,x^i)\to\max$$

$$\sum_{i=1}^{n}x^i=y,$$

$$x_i\geq 0.$$

Is considered by some $\tilde{\lambda}_i$ and f and $\tilde{\lambda}_i(f)$ is chosen supposing that it is possible. Then

$$\beta_i=\frac{\tilde{\lambda}_i(f)}{U^i(f,\tilde{x}(f))},$$

where $\beta_i=\dfrac{1}{\gamma_i}$.

The function is introduced

$$r_i(f,x^i)=\beta_i\nabla_x U^i(f,x^i).$$

Then it is supposed that $q_1=\beta_1 t,\ q_i=\beta_i\bar{f}^i\quad (i=\overline{2,n}),\ \ C_i=q_i A_i\quad (i=\overline{1,n})$, where the element a_{ke}^i of the matrix A_i has the following form

$$a_{ke}^{i} = \frac{\partial^2 F^i}{\partial x_k^i \partial x_e^i}(\tilde{x}^i)$$
.

Some statements are given below concerning the matrix C_i.

Lemma 1. [1] B y $n > 2$ the matrix C_i $(i = \overline{1,n})$ is invertible and

$$C_i^{-1} = \frac{(C_{ke}^i)_{k,l=1}^n}{2(n-2)a_{12}^i q_i x_1^i x_2^i},$$

where

$$C_{kl}^i = \begin{cases} x_k^i x_l^i, \ (l \neq k) \\ (3-n)(x_k^i)^2, (l = k) \end{cases}$$

Consequence 1.For $n = 3$ on the main diagonal of the matrix C_i^{-1} lay only zeros.

Consequence2. For $n > 3$ the elements on the main diagonal of the matrix C_i^{-1} are positive, and all other elements are negative.

Main results

Let us consider the case $n = 3$ in detail. Then the matrices C_i^{-1} take the forms

$$C_i^{-1} = \frac{1}{2q_i a_{12}^i}\begin{pmatrix} 0 & 1 & \frac{x_3^i}{x_1^i} \\ 1 & 0 & \frac{x_3^i}{x_1^i} \\ \frac{x_3^i}{x_2^i} & \frac{x_3^i}{x_1^i} & 0 \end{pmatrix}$$

and also is valid

$$y^1 = -C_1^{-1}K^{-1}\left[\left(C_2^{-1} + C_3^{-1}\right)\beta_1 \frac{\partial F^1}{\partial x^1} + \left(C_2^{-1} + C_3^{-1}\right)\overline{\delta}_1 + C_3 \overline{\delta}_2\right],$$

$$y^2 = C_2^{-1}K^{-1}\left[C_1^{-1}\beta_1 \frac{\partial F^1}{\partial x^1} - C_1^{-1}\overline{\delta}_1 - C_3^{-1}\overline{\delta}_2\right],$$

$$y^3 = C_3^{-1}K^{-1}\left[C_1^{-1}\beta_1 \frac{\partial F^1}{\partial x^1} + C_1^{-1}\overline{\delta}_1 + \left(C_1^{-1} - C_2^{-1}\right)\overline{\delta}_2\right].$$

From these relations it is easy to deduce the following statement.

Let

$$\tilde{y}^1 = -C_1^{-1}K^{-1}\left(C_2^{-1} + C_3^{-1}\right)\beta_1 \frac{\partial F^1}{\partial x^1}, \quad \tilde{y}^i = -C_i^{-1}K^{-1}C_1^{-1}\beta_1 \frac{\partial F^1}{\partial x^1} \quad (i = 2,3).$$

Lemma 2. If $\delta_1 > 0,\ \delta_2 = 0,$ then the signs of the coordinates of the vectors $y^i \ (i = 1,2,3)$ coinside with the signs of the coordinates of the corresponding vectors $\tilde{y}^i$ (i=1,2,3)

Lemma3. Let $\delta_1 > 0,\ (i = 1,2)$. Then the signs of the coordinates of the vectors y^1 and y^3 coinside with the signs of the coordinates of the corresponding vectors $\bar{y}^1$ and $\bar{y}^3$

The signs of the coordinates of the vectors y^2 may differ from the signs of the coordinates of the vectors $\tilde{y}^3$.

One can immediately check the validity of the following statement.

Theorem 1.The signs and values of the first coordinates of the vectors y^1, y^2, y^3

are completely defined by the matrix $C_i^{-1} \ (i = 1,2,3)$, number β_1 and vector $\frac{\partial F^1}{\partial x^1}$ and do not depend on δ_1 and δ_2

Note. By $n > 3$ the similar theorem is not true.

It presents interest the norm of the growth rate of each sector separately. To estimate this norm some facts are required. To prove the statement Metzler matrix [3] $C = \left(C_{ij}\right)_{i,j=1}^n$ will be used i.e. the matrix for which $C_{ij} \geq 0 \ (i \neq j)$. Let us sketch the proof of the follow in lemma.

Lemma 4.Let C be a symmetric Metzler matrix. Then it is valid the inequality

$$\|C\| \leq \|A\| + \mu$$

Really, for the Metzler matrix C it is satisfied $A = C + \mu I$,, where μ is some positive number, $A \geq 0$. Note that in this case the matrix A will be symmetric.

Let $\rho(C)$ –be spectral radius for the matrix C. Then

$$\rho(C) = \rho(A - \mu I) \le \rho(A) + \rho(-\mu I) = \rho(A) + \mu\rho(I) = \rho(A) + \mu..$$

From this and the fact that the norm of the symmetric operator coincide with the spectral radius it follows the proof of the inequality.

Lemma 5.Let the matrix $C = \left(C_{ij}\right)_{i,j=1}^{n}$ be given and

$$C_{ij} = \begin{cases} \dfrac{(3-n)x_i^2}{x_1 x_2}, & i = j \\ \dfrac{x_i x_j}{x_1 x_2}, & i \ne j, \end{cases}$$

Where x_i is a vector with positive coordinates. Then for the nonnegative matrix $A = C + \mu I$, where

$$\mu = (n-3)\max_{k=\overline{1,n}} \frac{x_k^2}{x_1 x_2}$$

and vector $x_0 = (x_1 x_2, \dots, x_1 x_2)$ is true

$$Ax_0 \le \varepsilon x_0 .$$

On the basis of the well-known theorem on the estimation of the spectral radius[4] we conclude that, ‖ A ‖ ≤ξ. This implies the estimate for the norm of the matrix C

$$\|C\| \le \xi + \mu .$$

From this inequality and from Lemma 2 [1] follows the estimate for the norm of the matrix C_i^{-1}

Let us consider the initial model.

Lemma 6. For $n > 3$ and $1 \le i \le n$ is valid the estimation

$$\left\|C_i^{-1}\right\| \le \frac{\xi_i + \mu_i}{2(n-2)\left|q_i a_{12}^i\right|},$$

where

$$\mu_i = (n-3)\max_{k=\overline{1,n}} \frac{\left(x_k^i\right)^2}{x_1^i x_2^i}, \quad \xi_i = \max_{k=\overline{1,n}} \sum_{j=1}^{n} a_{kj}^{-1}.$$

Here a_{kj}^{-i} are the elements of the matrix A^i, constructed similarly to A from Lemma 5.

<u>Theorem2.</u>For $y^k \quad (k=\overline{1,n})$ is valid the estimation

$$\left\|y^k\right\| \le \frac{\overline{L}^2}{\left(q_p a_{12}^p\right)^2}\left\|K^{-1}\right\|\left\{\beta_1\left\|\frac{\partial F^1}{\partial x^1}\right\| + \frac{\delta}{2}\left[k(k-1)+(n-k)(1+n-k)\right]\right\} \tag{1}$$

Proof. Substituting the relation for $B_{ki} \quad (i=\overline{1,n-1})$ into the formula (5) [1] and making some evident transformations we obtain the equality

$$y^k = -C_k^{-1}K^{-1}\left[-C_1^{-1}B_1\frac{\partial F^1}{\partial x^1} + (-C_1^{-1}\overline{\delta}_1 - (C_1^{-1}+C_2^{-1})\overline{\delta}_2) - \ldots\right.$$

$$\left.\ldots - \sum_{i=1}^{k-1} C_1^{-1}\overline{\delta}_{k-1} + \sum_{i=k+1}^{n} C_1^{-1}\overline{\delta}_{k+1} + \ldots + C_n^{-1}\overline{\delta}_{n-1}\right]$$

From this one can get the estimation

$$\left\|y^k\right\| \le \left\|C_k^{-1}\right\|\left\|K^{-1}\right\|\beta_1\left\|C_1^{-1}\right\|\left\|\frac{\partial F^1}{\partial x^1}\right\| + \delta[(k-1)]\left\|C_1^{-1}\right\| + (k-2)\left\|C_2^{-1}\right\| + \ldots$$

$$\ldots + \left\|C_{k+1}^{-1}\right\| + \ldots + (n-k-1)\left\|C_{n-1}^{-1}\right\| + (n-k)\left\|C_n^{-1}\right\|\big]\big\},$$

where $\delta = \max_{i=\overline{1,n}}\left|\delta_i\right|$.

Let $G = \max_{i=\overline{1,n}}\left\|C_i^{-1}\right\|$. Then we chose the elements x_1^i from the cones

$$K^i = \left\{(x_1^i,\ldots,x_n^i)\left|\left(\frac{x_p^i}{x_g}\right)_{p,g=\overline{1,n},\ p\ne g} < L_i\right.\right\}$$

and take

$$L = \max_{i=\overline{1,n}} L_i^2, \quad q_p\left|a_{12}^p\right| = \min_{i=\overline{1,n}} q_i\left|a_{12}^i\right|.$$

Thus for large enough r it is valid the estimation

$$G = \frac{\overline{L}}{q_p \left| a_{12}^p \right|} ,$$

where $\overline{L} = const$, therefore (1) is valid.

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