ON A SYSTEM OF DİFFERENCE EQUATIONS

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ABSTRACT

In this paper, I investigated the solutions of the system of the difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}-1},\quad y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}-1},\quad z_{n+1}=\frac{x_n}{y_n}$$

where $y_n x_{n-1}\neq 1$, $x_n y_{n-1}\neq 1$ and $x_0,\ x_{-1},\ y_0,\ y_{-1}\neq 0,\ z_0,\ z_{-1}\in \mathbb{R}$.

Keywords: Difference equations, difference equations systems, solutions, equilibrium point, behavior of solutions, rational difference equations, systems of rational difference equations.

INTRODUCTION

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is a necessity for some techniques which can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology etc. There are many papers with related to the difference equations system for example,

In [1] A. S. Kurbanlı, C. Çinar and I. Yalcinkaya, studied the behavaior of positive solutions of the system of rational difference equations

$$x_{n+1}=\frac{x_{n-1}}{y_n x_{n-1}+1},\quad y_{n+1}=\frac{y_{n-1}}{x_n y_{n-1}+1}.$$

In [11] Cinar studied the solutions of the systems of difference equations

$$x_{n+1}=\frac{1}{y_n},\quad y_{n+1}=\frac{y_n}{x_{n-1}y_{n-1}}.$$

In [25] Gamouzis and Papaschinnopoulos studied the global asymptotic behavaior of positive solutions of the systems of rational difference equations

$$x_{n+1}=1+\frac{x_n}{y_{m-n}},\quad y_{n+1}=1+\frac{y_n}{x_{n-m}}$$

In [28, 29] Özban studied the positive solutions of the system of rational difference equations

$$x_n=\frac{a}{y_{n-3}},\quad y_n=\frac{by_{n-3}}{x_{n-q}y_{n-q}}$$

and

$$x_{n+1}=1+\frac{x_n}{y_{n-k}},\quad y_{n+1}=1+\frac{y_n}{x_{n-m}y_{n-m-k}}$$

In [35, 36] Clark and Kulenovic investigate the global asmptotic stability

$$x_{n+1}=\frac{x_n}{a+cy_n},\quad y_{n+1}=\frac{y_n}{b+dx_n}$$

Also see reference.

In this paper, I investigated the behavaior of the positive solutions of the difference equations system

(1.1) $$x_{n+1} = \frac{x_{n-1}}{y_n x_{n-1} - 1},\ y_{n+1} = \frac{y_{n-1}}{x_n y_{n-1} - 1},\ z_{n+1} = \frac{x_n}{y_n}$$

where $y_n x_{n-1} \neq 1$, $x_n y_{n-1} \neq 1$ and $x_0, x_{-1}, y_0, y_{-1} \neq 0, z_0, z_{-1} \in \mathbb{R}$ non-zero arbitrary real numbers.

MAIN RESULTS

Theorem 1. Let
$y_0 = a \neq 0,\ y_{-1} = b \neq 0,\ x_0 = c,\ x_{-1} = d$
be arbitrary real numbers and let $\{x_n, y_n\}$ be a solution of the system (1.1). Also, assume that $ad \neq 1$ and $cb \neq 1$. Then all solutions of (1.1) are

(1.2) $$x_n = \begin{cases} \dfrac{d}{(ad-1)^n}, & n-odd \\ c(cb-1)^n & n-even \end{cases}$$

(1.3) $$y_n = \begin{cases} \dfrac{b}{(cb-1)^n}, & n-odd \\ a(ad-1)^n & n-even \end{cases}$$

(1.4) $$z_n = \begin{cases} \dfrac{c(cb-1)^{n-1}}{a(ad-1)^{n-1}}, & n-odd \\ \dfrac{d(cb-1)^n}{b(ad-1)^n} & n-even \end{cases}$$

Proof. For $n = 0,1,2,3$ I have

$$x_1 = \frac{x_{-1}}{y_0 x_{-1} + 1} = \frac{d}{ad-1}$$

$$y_1 = \frac{y_{-1}}{x_0 y_{-1} - 1} = \frac{b}{cb-1}$$

$$z_1 = \frac{x_0}{y_0} = \frac{c}{a}$$

$$x_2 = \frac{x_0}{y_1 x_0 - 1} = \frac{c}{\left(\frac{b}{cb-1}\right)c - 1} = c(cb-1)$$

$$y_2 = \frac{y_0}{x_1 y_0 - 1} = \frac{a}{\frac{d}{ad-1}a - 1} = a(ad-1)$$

$$z_2=\frac{x_1}{y_1}=\frac{\frac{d}{ad-1}}{\frac{b}{cb-1}}=\frac{d(cb-1)}{b(ad-1)}$$

$$x_3=\frac{x_1}{y_2x_1-1}=\frac{\frac{d}{ad-1}}{a(ad-1)\left(\frac{d}{ad-1}\right)-1}=\frac{d}{(ad-1)^2}$$

$$y_3=\frac{y_1}{x_2y_1-1}=\frac{\frac{b}{cb-1}}{c(cb-1)\frac{b}{cb-1}-1}=\frac{b}{(cb-1)^2}$$

$$z_3=\frac{x_2}{y_2}=\frac{c(cb-1)}{a(ad-1)}$$

for $n=k$ assume that

$$x_{2k-1}=\frac{d}{(ad-1)^k}$$

$$x_{2k}=c(cb-1)^k$$

$$y_{2k-1}=\frac{b}{(cb-1)^k}$$

$$y_{2k}=a(ad-1)^k$$

and

$$z_{2k-1}=\frac{c(cb-1)^{k-1}}{a(ad-1)^{k-1}}$$

$$z_{2k}=\frac{d(cb-1)^k}{b(ad-1)^k}$$

are true. Then for $n=k+1$ I will show that (1.2), (1.3) and (1.4) are true.
From (1.1), whe have

$$x_{2k+1}=\frac{x_{2k-1}}{y_{2k}x_{2k-1}-1}=\frac{\frac{d}{(ad-1)^k}}{a(ad-1)^k\frac{d}{(ad-1)^k}-1}=\frac{d}{(ad-1)^{k+1}}$$

$$y_{2k+1}=\frac{y_{2k-1}}{x_{2k}y_{2k-1}-1}=\frac{\frac{b}{(cb-1)^k}}{c(cb-1)^k\frac{b}{(cb-1)^k}-1}=\frac{b}{(cb-1)^{k+1}}$$

Also, smilarly from (1.1), I have

$$z_{2k+1}=\frac{c(cb-1)^k}{a(ad-1)^k}$$

Also, I have

$$x_{2k+2}=\frac{x_{2k}}{y_{2k+1}x_{2k}-1}=\frac{c(cb-1)^k}{\frac{b}{(cb-1)^{k+1}}c(cb-1)^k-1}=\frac{c(cb-1)^k}{\frac{b}{(cb-1)}c-1}=c(cb-1)^{k+1}$$

$$y_{2k+2}=\frac{y_{2k}}{x_{2k+1}y_{2k}-1}=\frac{a(ad-1)^k}{\frac{d}{(ad-1)^{k+1}}a(ad-1)^k-1}=\frac{a(ad-1)^k}{\frac{d}{(ad-1)}-1}=a(ad-1)^{k+1}$$

and

$$z_{2k+2}=\frac{x_{2k+1}}{y_{2k+1}}=\frac{\frac{d}{(ad-1)^{k+1}}}{\frac{b}{(cb-1)^{k+1}}}=\frac{d(cb-1)^{k+1}}{b(ad-1)^{k+1}}.$$

Corollary 1. Let a, b, c, d be arbitrary real numbers and let $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $0 < a, b, c, d < 1$ then I have

$$\lim_{n\to\infty} x_{2n-1}=\lim_{n\to\infty} y_{2n-1}=\infty,$$

$$\lim_{n\to\infty} z_{2n-1}=\begin{cases}0, & cb<ad\\ \infty, & cb>ad\\ \frac{c}{a}, & cb=ad\end{cases}$$

and

$$\lim_{n\to\infty} x_{2n}=\lim_{n\to\infty} y_{2n}=0,$$

$$\lim_{n\to\infty} z_{2n}=\begin{cases}0, & cb<ad\\ \infty, & cb>ad\\ \frac{d}{b}, & cb=ad\end{cases}$$

Proof. From $0 < a, b, c, d < 1$ I have $0<ad<1\Rightarrow -1<ad-1<0$ and $0<cb<1\Rightarrow -1<cb-1<0$. Hence I obtain

$$\lim_{n\to\infty} x_{2n-1}=\lim_{n\to\infty}\frac{d}{(ad-1)^n}=d\lim_{n\to\infty}\frac{1}{(ad-1)^n}=\begin{cases}-\infty, & n-odd\\ \infty, & n-even\end{cases},$$

$$\lim_{n\to\infty} y_{2n-1}=\lim_{n\to\infty}\frac{b}{(cb-1)^n}=b\lim_{n\to\infty}\frac{1}{(cb-1)^n}=\begin{cases}-\infty, & n-odd\\ \infty, & n-even\end{cases}$$

and

$$\lim_{n\to\infty} z_{2n-1}=\lim_{n\to\infty}\frac{c(cb-1)^{n-1}}{a(ad-1)^{n-1}}=\frac{c}{a}\lim_{n\to\infty}\left(\frac{cb-1}{ad-1}\right)^{n-1}$$

$$=\begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{a}, & cb = ad \end{cases}.$$

Similarly, I have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c\left(cd-1\right)^n = c\lim_{n\to\infty}\left(cd-1\right)^n = c.0 = 0$$
,
$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a\left(af-1\right)^n = a\lim_{n\to\infty}\left(af-1\right)^n = a.0 = 0$$

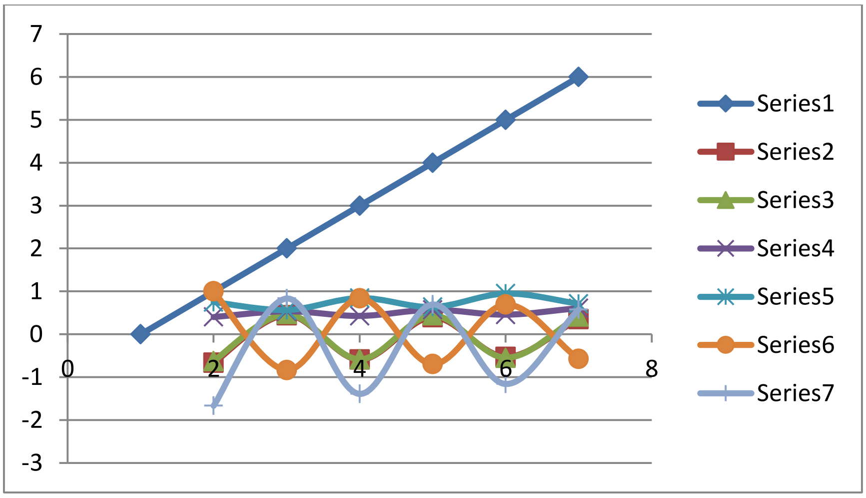
and

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} \frac{d\left(cb-1\right)^n}{b\left(ad-1\right)^n} = \frac{d}{b}\lim_{n\to\infty}\left(\frac{cb-1}{ad-1}\right)^n$$

$$=\begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{b}, & cb = ad \end{cases}.$$

Example 1. If $y_0 = 1/2,\ y_{-1} = 1/2,\ x_0 = 1/2,\ x_{-1} = 1/2$ then the solutions of Eq. (1.1) can be represented by the following table:

i	1	2	3	4	5	6
x_{2i-1}	-0.66667	0.44444	-0.59259	0.39506	-0.52675	0.35117
x_{2i}	-0.62500	0.46875	-0.58594	0.43945	-0.54932	0.41199
y_{2i-1}	0.40000	0.53333	0.42667	0.56889	0.45511	0.60681
y_{2i}	0.75000	0.56250	0.84375	0.63281	0.94922	0.71191
z_{2i-1}	1	-0.83333	0.83333	-0.69444	0.69444	-0.57870
z_{2i}	-1.66667	0.83333	-1.38889	0.69444	-1.15741	0.57870



Corollary 2. Let a, b, c, d be arbitrary real numbers and let $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $1 < ad, cb < 2$ then I have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = \infty,$$

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{c}{a}, & cb = ad \end{cases}$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = 0,$$

$$\lim_{n\to\infty} z_{2n} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{d}{b}, & cb = ad \end{cases}$$

Proof. From

$1 < ad < 2 \Rightarrow 0 < ad - 1 < 1$ we have $\lim_{n\to\infty} (ad-1)^n = 0$

from $1 < cb < 2 \Rightarrow 0 < cb - 1 < 1$ I have $\lim_{n\to\infty} (cd-1)^n = 0$

Hence I have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = \begin{cases} -\infty, & d < 0 \\ \infty, & d > 0 \end{cases},$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b \lim_{n\to\infty} \frac{1}{(cb-1)^n} = \begin{cases} -\infty, & b < 0 \\ \infty, & b > 0 \end{cases}$$

and

$$\lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} \frac{c(cb-1)^{n-1}}{a(ad-1)^{n-1}} = \frac{c}{a} \lim_{n\to\infty} \left(\frac{cb-1}{ad-1} \right)^{n-1}$$

$$= \begin{cases} ad > cb \Rightarrow ad - 1 > cb - 1 \Rightarrow \frac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad - 1 < cb - 1 \Rightarrow \frac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad - 1 = cb - 1 \Rightarrow \frac{cb-1}{ad-1} = 1 \end{cases} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{c}{a}, & cb = ad \end{cases}.$$

Similarly, I have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cd-1)^n = c \lim_{n\to\infty} (cd-1)^n = c.0 = 0,$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(af-1)^n = a \lim_{n\to\infty} (af-1)^n = a.0 = 0$$

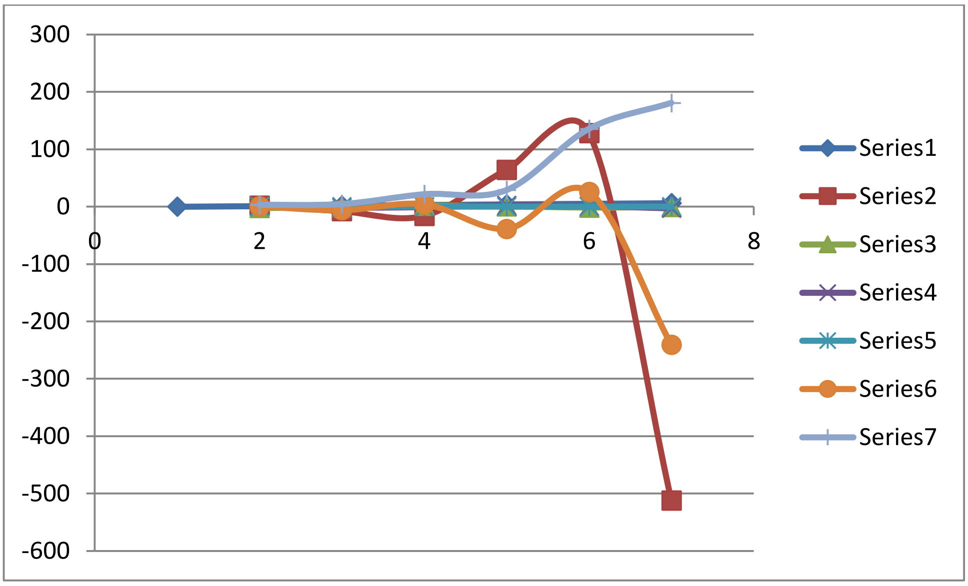
and

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} \frac{d(cb-1)^n}{b(ad-1)^n} = \frac{d}{b} \lim_{n\to\infty} \left(\frac{cb-1}{ad-1} \right)^n$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{b}, & cb = ad \end{cases}.$$

Example 2. If $y_0 = 3/2,\ y_{-1} = 4/3,\ x_0 = 1,\ x_{-1} = 1$ then the solutions of Eq. (1.1) can be represented by the following table:

i	1	2	3	4	5	6
x_{2i-1}	2	-8	-16	64	128	-512
x_{2i}	-2.33333	-0.77778	1.81481	0.60494	-1.41152	-0.47051
y_{2i-1}	0.57143	-1.71429	-0.73469	2.20408	0.94461	-2.83382
y_{2i}	0.37500	-0.18750	-0.04688	0.02344	0.00586	-0.00293
z_{2i-1}	0.66667	-6.22222	4.14815	-38.71605	25.81070	-240.89986
z_{2i}	3.50000	4.66667	21.77778	29.03704	135.50617	180.67490



Corollary 3. Let $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $cb,\ ad \in (-\infty, 0)$ then

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = 0$$

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{a}, & cb = ad \end{cases}$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = \infty$$

$$\lim_{n\to\infty} z_{2n} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{b}, & cb = ad \end{cases}$$

Proof. From

$-\infty < cb,\ ad < 0 \Rightarrow -\infty < cb-1,\ ad-1 < -1$

I have

$$\lim_{n\to\infty} (cb-1)^n = \begin{cases} -\infty, & n-odd \\ \infty, & n-even \end{cases}$$

and

$$\lim_{n\to\infty} (ad-1)^n = \begin{cases} -\infty, & n-odd \\ \infty, & n-even \end{cases}$$

Hence, I have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d \lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.0 = 0,$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b \lim_{n\to\infty} \frac{1}{(cb-1)^n} = b.0 = 0$$

and

$$\lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} \frac{c(cb-1)^{n-1}}{a(ad-1)^{n-1}} = \frac{c}{a} \lim_{n\to\infty} \left(\frac{cb-1}{ad-1}\right)^{n-1}$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{a}, & cb = ad \end{cases}.$$

Similarly, I have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cd-1)^n = c \lim_{n\to\infty} (cd-1)^n = c.\infty = \begin{cases} -\infty, & c > 0 \text{ and } n-odd \\ +\infty, & c < 0 \text{ and } n-odd \\ +\infty, & c > 0 \text{ and } n-even \\ -\infty, & c < 0 \text{ and } n-even \end{cases},$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a \lim_{n\to\infty} (ad-1)^n = a.\infty = \begin{cases} -\infty, & a > 0 \text{ and } n-odd \\ +\infty, & a < 0 \text{ and } n-odd \\ +\infty, & a > 0 \text{ and } n-even \\ -\infty, & a < 0 \text{ and } n-even \end{cases}$$

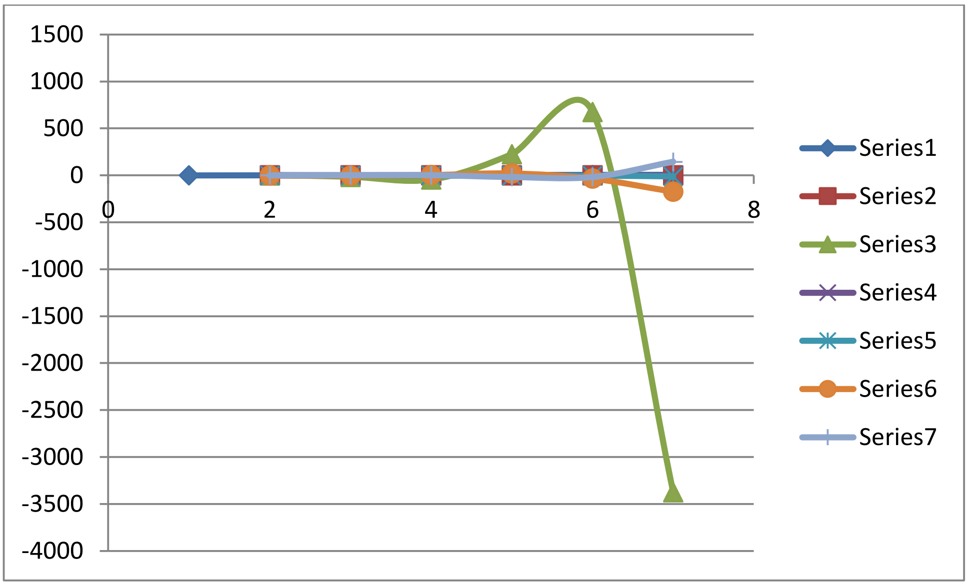
and

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} \frac{d(cb-1)^n}{b(ad-1)^n} = \frac{d}{b} \lim_{n\to\infty} \left(\frac{cb-1}{ad-1}\right)^n$$

$$= \begin{cases} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \dfrac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \dfrac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \dfrac{cb-1}{ad-1} = 1 \end{cases} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{d}{b}, & cb = ad \end{cases}.$$

Example 3. If $y_0 = -2,\ y_{-1} = -4,\ x_0 = 1,\ x_{-1} = 1$ then the solutions of Eq. (1.1) can be represented by the following table:

i	1	2	3	4	5	6
x_{2i-1}	-0.33333	0.55556	-0.18519	0.30864	-0.10288	0.17147
x_{2i}	3.00000	-15.00000	-45.00000	225.00000	675.00000	-3375.00001
y_{2i-1}	1.33333	0.26667	-0.08889	-0.01778	0.00593	0.00119
y_{2i}	-1.20000	-3.59999	-2.16000	-6.47999	-3.88800	-11.66400
z_{2i-1}	-0.50000	-2.50000	4.16667	20.83333	-34.72222	-173.61111
z_{2i}	-0.25000	2.08333	2.08333	-17.36111	-17.36111	144.67593



Corollary 4. Let $\{x_n, y_n, z_n\}$ be a solution of the system (1.1). If $a,b,c,d \in \mathbb{R}$ and $ad, cb \in (2,+\infty)$ then I have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} y_{2n-1} = 0$$

$$\lim_{n\to\infty} z_{2n-1} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \dfrac{c}{a}, & cb = ad \end{cases}$$

and

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} y_{2n} = \infty$$

$$\lim_{n\to\infty} z_{2n} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{d}{b}, & cb = ad \end{cases}$$

Proof. From

$2 < cb,\ ad < +\infty \Rightarrow 1 < cb-1,\ ad-1 < +\infty$

I have

$$\lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{d}{(ad-1)^n} = d\lim_{n\to\infty} \frac{1}{(ad-1)^n} = d.0 = 0,$$

$$\lim_{n\to\infty} y_{2n-1} = \lim_{n\to\infty} \frac{b}{(cb-1)^n} = b\lim_{n\to\infty} \frac{1}{(cb-1)^n} = b.0 = 0$$

and

$$\lim_{n\to\infty} z_{2n-1} = \lim_{n\to\infty} \frac{c(cb-1)^{n-1}}{a(ad-1)^{n-1}} = \frac{c}{a}\lim_{n\to\infty}\left(\frac{cb-1}{ad-1}\right)^{n-1}$$

$$= \begin{Bmatrix} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \frac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \frac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \frac{cb-1}{ad-1} = 1 \end{Bmatrix} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{c}{a}, & cb = ad \end{cases}.$$

Similarly, I have

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} c(cd-1)^n = c\lim_{n\to\infty}(cd-1)^n = c.\infty = \begin{cases} -\infty, & c > 0 \\ +\infty, & c < 0 \end{cases},$$

$$\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} a(ad-1)^n = a\lim_{n\to\infty}(ad-1)^n = a.\infty = \begin{cases} -\infty, & a > 0 \\ +\infty, & a < 0 \end{cases}$$

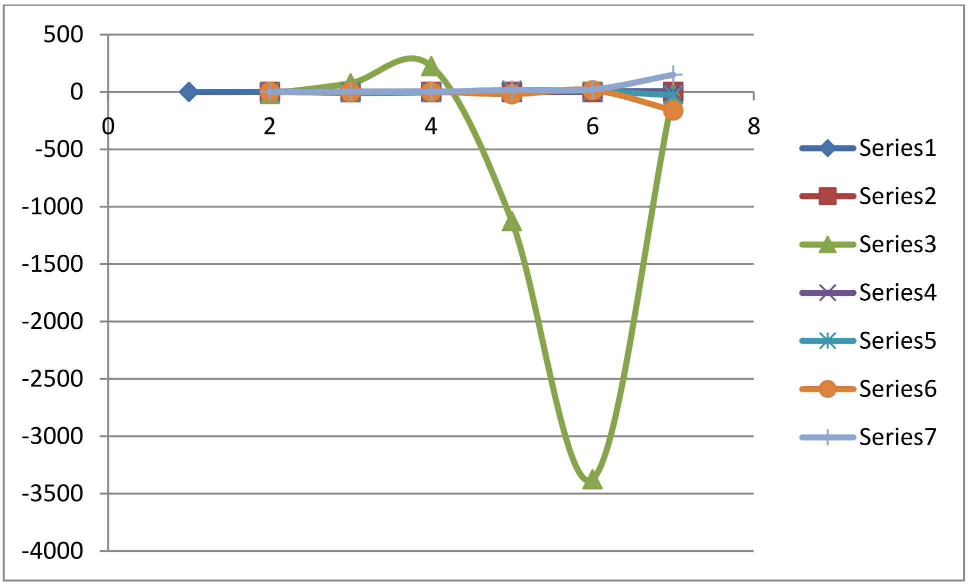
and

$$\lim_{n\to\infty} z_{2n} = \lim_{n\to\infty} \frac{d(cb-1)^n}{b(ad-1)^n} = \frac{d}{b}\lim_{n\to\infty}\left(\frac{cb-1}{ad-1}\right)^n$$

$$= \begin{Bmatrix} ad > cb \Rightarrow ad-1 > cb-1 \Rightarrow \frac{cb-1}{ad-1} < 1 \\ ad < cb \Rightarrow ad-1 < cb-1 \Rightarrow \frac{cb-1}{ad-1} > 1 \\ ad = cb \Rightarrow ad-1 = cb-1 \Rightarrow \frac{cb-1}{ad-1} = 1 \end{Bmatrix} = \begin{cases} 0, & cb < ad \\ \infty, & cb > ad \\ \frac{d}{b}, & cb = ad \end{cases}.$$

Example 4. If $y_0 = 5,\ y_{-1} = 4,\ x_0 = 1,\ x_{-1} = 1$ then the solutions of Eq. (1.1) can be represented by the following table:

i	1	2	3	4	5	6
x_{2i-1}	0.25000	-0.56250	-0.14063	0.31641	0.07910	-0.17798
x_{2i}	-15.00000	75.00000	225.00000	-1125.000000	-3375.00000	-15.00000
y_{2i-1}	0.80000	-0.26667	-0.05333	0.01778	0.00356	-0.00119
y_{2i}	2.22222	-8.88889	-3.95062	15.80247	7.02332	-28.09328
z_{2i-1}	0.20000	-2.25000	1.68750	-18.98437	14.23828	-160.18066
z_{2i}	0.31250	2.10937	2.63672	17.79785	22.24731	150.16937



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