NEW IMPROVED NEWTON METHOD WITH (*K*+2) ORDER CONVERGENCE FOR SOLVING QUADRATIC EQUATIONS

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ABSTRACT

The objective of this paper is to define a new improved Newton method for finding simple roots of quadratic equations. It is proved that the new two-point method has the convergence order of $(k+2)$ requiring only two function evaluations per iteration, where k is the number of terms in the generating series. It is observed that our proposed method is very simple to construct when compare to the Babajee's and Ahmad's two-point method.

Keywords: Newton method; Quadratic equations; Kung-Traub's conjecture; Efficiency index; Optimal order of convergence.
Subject Classifications: AMS (MOS): 65H05.

INTRODUCTION

In this paper, we present a new two-point $(k+2)$-order iterative method to find a simple root α of the nonlinear equation $f(x)=0$, where $f:D\subset\Re\to\Re$ for an open interval D is a scalar function. Many higher order multi-point variants of the Newton method have been developed based on the Kung and Traub conjecture [5]. Here we present a new iterative method which has a better efficiency index than the classical Newton method [4,6,7,8] and is equivalent to the Ahmad [1,2] and Babajee [3] method. We have found that the Ahmad and Babajee have presented similar two-point $(k+2)$-order iterative method and the main difference between the proposed method and the method given in [1-3] is that the proposed method is based on a different weight function and is much simpler to construct.

It is well known that the classical Newton method for finding simple roots is given by

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which converges quadratically [4,6,7,8]. For the purpose of this paper, we improve the classical Newton method and to construct a new $(k+2)$-order iterative method for finding simple roots of quadratic equations. The new method presented in this paper only uses three evaluations of the function per iteration. Kung and Traub conjectured that the multipoint iteration methods, without memory based on n evaluations, could achieve optimal convergence order 2^{n-1}. In fact, we have obtained a higher order of convergence than the maximum order of convergence suggested by Kung and Traub conjecture [5]. We demonstrate that the Kung and Traub conjecture fails for a particular case.

PRELIMINARIES

Definition 1 Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence p is given by

$$\lim_{n \to \infty} \frac{x_{n+1} - \alpha}{(x_n - \alpha)^p} = \lambda \neq 0 \quad (2)$$

where $p \in \Re^+$ and λ is the asymptotic error constant. Let $e_n = x_n - \alpha$ be the error in the *n*th iteration, then the relation

$$e_{n+1} = \zeta e_n^p + O\left(e_n^{p+1}\right), \quad (3)$$

is the error equation. If the error equation exists, then *p* is the order of convergence of the iterative method, [4,6,7,8].

Definition 2 Let *n* be the number of function evaluations of the iterative method. The efficiency of the iterative method is measured by the concept of efficiency index and defined as

$$E(n, p) = \sqrt[n]{p} \quad (4)$$

where *p* is the order of convergence of the method, [7].

Definition 3 (Kung and Traub conjecture) Let $x_{n+1} = g(x_n)$ define as an iterative function without memory with *n*-evaluations. Then

$$p(g) \leq p_{opt} = 2^{n-1}, \quad (5)$$

where p_{opt} is the maximum order, [5].

CONSTRUCTION OF THE NEW METHOD AND ANALYSIS OF CONVERGENCE

In this section we define a new class of two-point $(k+2)$-order method for finding simple roots of a quadratic equation. In fact, the new iterative method is an improvement of the classical Newton method, given by (1). We shall demonstrate that the new two-point method can be constructed to produce any desired order of convergence and is equivalent to the Ahmad [1,2] and Babajee method [3]. The order of convergence the new iterative method is determined by the *k*, number of terms in the generating series, which improves the classical Newton method. First, we will state the essentials of the Ahmad and Babajee method, hence we will illustrate the equivalency of the new method.

Ahmad and Babajee presented a two-point $(k+2)$-order method in [1-3] and is expressed by

$$y_n = x_n - \left(\frac{2}{3}\right) u(x_n) \quad (6)$$

$$x_{n+1} = x_n - u(x) H(\tau, k) \quad (7)$$

where

$$H(\tau, k) = 1 + \sum_{i=0}^{k-1} a_i (\tau - 1)^i \quad u(x) = \frac{f(x_n)}{f'(x_n)} \text{ and } \tau = \frac{f'(y_n)}{f'(x_n)}, \quad (8)$$

The first six constants coefficients of the terms of (11) are

$$a_1 = \frac{3}{4}, \quad a_2 = \frac{9}{8}, \quad a_3 = \frac{135}{64}, \quad a_4 = \frac{567}{128}, \quad a_5 = \frac{5103}{512}, \quad a_6 = \frac{24057}{1024}, \quad (9)$$

and the coefficients a_i is calculated by

$$a_{k+1} = C_k \left(-\frac{3}{4}\right)^{k+1} \quad (10)$$

where c_k is the asymptotic error constant, obtained by the previous order of the iterative method [3]. The few members of (7) with their error equation are

1. $k = 1$: Two-point third-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 - \left(\tfrac{3}{4}\right)(\tau - 1)\right] \tag{11}$$

and the error equation

$$e_{n+1} - \alpha = 2c_2^2 e_n^3. \tag{12}$$

2. $k = 2$: Two-point fourth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 - \left(\tfrac{3}{4}\right)(\tau - 1) + \left(\tfrac{9}{8}\right)(\tau - 1)^2\right] \tag{13}$$

and the error equation

$$e_{n+1} - \alpha = 5c_2^3 e_n^4. \tag{14}$$

2. $k = 3$: Two-point fifth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 - \left(\tfrac{3}{4}\right)(\tau - 1) + \left(\tfrac{9}{8}\right)(\tau - 1)^2 - \left(\tfrac{135}{64}\right)(\tau - 1)^3\right] \tag{15}$$

and the error equation

$$e_{n+1} - \alpha = 14c_2^4 e_n^5. \tag{16}$$

3. $k = 4$: Two-point sixth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 - \left(\tfrac{3}{4}\right)(\tau - 1) + \left(\tfrac{9}{8}\right)(\tau - 1)^2 - \left(\tfrac{135}{64}\right)(\tau - 1)^3 + \left(\tfrac{567}{128}\right)(\tau - 1)^4\right] \tag{17}$$

and the error equation

$$e_{n+1} - \alpha = 42c_2^5 e_n^6. \tag{18}$$

4. $k = 5$: Two-point seventh-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 - \left(\tfrac{3}{4}\right)(\tau - 1) + \left(\tfrac{9}{8}\right)(\tau - 1)^2 - \left(\tfrac{135}{64}\right)(\tau - 1)^3 + \left(\tfrac{567}{128}\right)(\tau - 1)^4 - \left(\tfrac{5103}{512}\right)(\tau - 1)^5\right] \tag{19}$$

and the error equation

$$e_{n+1} - \alpha = 132c_2^6 e_n^7. \tag{20}$$

The Method

The new two-point $(k+2)$-order Newton-type method is expressed by

$$y_n = x_n - u(x_n) \tag{21}$$

$$x_{n+1} = x_n - u(x)G(z,k) \tag{22}$$

where

$$G(z,k) = 1 + \sum_{i=1}^{k} a_i z^i \tag{23}$$

$$u(x_n) = \frac{f(x_n)}{f'(x_n)} \quad \text{and} \quad z = \frac{f(y_n)}{f(x_n)}, \tag{24}$$

The first eight constants coefficients of the terms of (22) are

$$a_1 = 1,\quad a_2 = 2,\quad a_3 = 5,\quad a_4 = 14,\quad a_5 = 42,\quad a_6 = 132,\quad a_7 = 429,\quad a_8 = 1430 \tag{25}$$

where x_0 is the initial guess and provided that denominators of (22) are not equal to zero. Now, we shall verify the convergence property of the new two-point $(k+2)$-order iterative method (22).

Theorem 1
Let $\alpha \in D$ be a simple zero of a sufficiently smooth function $f : D \subset \Re \to \Re$ for an open interval *D*. If the initial guess x_0 is sufficiently close to α, then the convergence order of the new two-point iterative method defined by (22) is $(k+2)$.

Proof
Let α be a simple root of $f(x)$, i.e. $f(\alpha)=0$ and $f'(\alpha)\neq 0$, and the error is expressed as $e = x - \alpha$.
The Taylor series expansion and taking into account $f(\alpha)=0$, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2 e_n^2\right). \tag{26}$$

$$f'(x_n) = f'(\alpha)(1+2c_2 e_n). \tag{27}$$

where

$$c_2 = \frac{f''(\alpha)}{f'(\alpha)} . \tag{28}$$

Dividing (14) by (15), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2c_2^2 e_n^3 - 4c_2^3 e_n^4 + 8c_2^3 e_n^4 \cdots . \tag{29}$$

and

$$\left(\frac{f(y_n)}{f(x_n)}\right) = c_2 e_n - 3c_2^2 e_n^2 + 8c_2^3 e_n^3 - 20c_2^4 e_n^4 + 48c_2^5 e_n^5 - \cdots . \tag{30}$$

Substituting (29) in (22), we obtain

$$e_{n+1} = e_n - \frac{f(x_n)}{f'(x_n)}, \tag{31}$$

$$AEC(1) = c_2 e_n^2. \tag{32}$$

It is well known that (32) is the asymptotic error constant for the classical Newton method defined by (1). Therefore, we take the coefficient of the error equation (32) as our first coefficient of the generating series given in (23) as $a_1 = 1$. Furthermore, we obtain a family of higher iterative method by increasing the terms of summation series of (23). Hence, we show the asymptotic error constant $AEC(k+1)$ for the $(k+2)$-order Newton-type method. The next seven members of (11) with their error equation are

1. $k=1$: Two-point third-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)[1+z] \tag{33}$$

and the error equation

$$AEC(2) = 2c_2^2 e_n^3. \tag{34}$$

2. $k=2$: Two-point fourth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1+z+2z^2\right] \tag{35}$$

and the error equation

$$AEC(3) = 5c_2^3 e_n^4. \tag{36}$$

3. $k=3$: Two-point fifth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 + z + 2z^2 + 5z^3\right] \tag{37}$$

and the error equation

$$AEC(4) = 14c_2^4 e_n^5. \tag{38}$$

4. $k=4$: Two-point sixth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 + z + 2z^2 + 5z^3 + 14z^4\right] \tag{39}$$

and the error equation

$$AEC(5) = 42c_2^5 e_n^6. \tag{40}$$

5. $k=5$: Two-point seventh-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5\right] \tag{41}$$

and the error equation

$$AEC(6) = 132c_2^6 e_n^7. \tag{42}$$

6. $k=6$: Two-point eighth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6\right] \tag{43}$$

and the error equation

$$AEC(7) = 429c_2^7 e_n^8. \tag{44}$$

7. $k=7$: Two-point ninth-order iterative method is given by

$$x_{n+1} = x_n - u(x_n)\left[1 + z + 2z^2 + 5z^3 + 14z^4 + 42z^5 + 132z^6 + 429z^7\right] \tag{45}$$

and the error equation

$$AEC(8) = 1430c_2^8 e_n^9. \tag{46}$$

It is well established that the maximum order of convergence of optimal methods with three functions evaluations is 4. As illustrated above we have obtained order of convergence greater than 4, hence the Kung and Traub conjecture fails for this particular case. To produce the next two-point higher order of convergence with only three function evaluations, we use the following principle. The process is very simple, using the coefficient of the error equation $AEC(k+1)$ as our coefficient a_k of the next term of the generating series, thus we can calculate the next higher order of convergence method. Hence, the coefficient a_k of the series (22) is obtained

$$a_k = \frac{AEC(k+1)}{c_2^k e_n^{k+1}} \tag{47}$$

where $k \geq 1$ and $AEC(k+1)$ is the error equation of the $(k+2)$-order Newton-type method.

Remark 1

The new two-point iterative method requires three function evaluations and has the order of convergence $(k+2)$. To determine the efficiency index of the new method, definition 2 shall be used. Hence, the efficiency index of the new iterative method given by (11) is $\sqrt[3]{k+2}$ and the efficiency index of the classical Newton method is $\sqrt[2]{2}$.

CONCLUSION

In this study, a new two-point $(k+2)$-order Newton-type method has been presented. The prime motive for presenting the new class of iterative method was to improve the two-point the Ahmad and Babajee method [1-3]. The main difference between the new method and the established method [1-3] is the weight function used in the generating series. We have found that the error equations of the new method and the Ahmad and Babajee method are identical. Furthermore, the coefficients of the generating series in the new method are obtained naturally, whereas the coefficients of the generating series in the Ahmad and Babajee method requires much more calculations. Therefore, it is evident that the coefficient in the generating series of the new method is much simpler than the Ahmad and Babajee method. The essential advantages of the new method are: very high computational efficiency; the new method is not limited to the Kung and Traub conjecture; better efficiency index than the classical Newton method. Finally, we conjecture that the proposed method may extended to higher order polynomials and be able to solve the systems of quadratic equations.

CONFLICT OF INTERESTS

The author declare that there is no conflict of interests regarding the publication of this paper.

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