A NEW DIRECTION FOR FINDING CONFIDENCE REGION OF MULTIPLE PARAMETERS

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ABSTRACT

There are three typical values of constant Gaussian curvature, 0, +1 and -1. All other values of the Gaussian curvature can be obtained by multiplying positive real numbers to these three typical values. The corresponding surfaces to these three typical values are cylinder, sphere and hyperboloid of one sheet. In this paper we will study the geometry of these surfaces and how to use them in a statistical confidence region.

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INTRODUCTION

The uniformization theorem states that every smooth Riemannian Surface, S, is conformally equivalent to a surface having constant curvature and the constant may be taken to be 0, +1 or -1. A surface of constant curvature, 0, is locally isometric to the Euclidean plane. A surface of constant curvature, +1, is locally isometric to the sphere. This means every point on the surface has an open neighborhood that is isometric to an open set on the unit sphere. Likewise, a surface of constant curvature -1 is locally isometric to the hyperboloid plane. In this paper we will study the area bounded by a geodesic line of a triangle or rectangular on these specialized surfaces. In the next section, we provide reasons to support our new direction. Then we derive the Gauss-Bonnet theorem and discuss the condition when a surface becomes a plane, sphere or hyperboloid. We found that the area of a geodesic triangle is proportional to the difference of the sum of the interior angle and π. The constant of proportionality is just the Gaussian curvature of the surface. Finally, we give a numerical example to show the difference between a classic confidence region and our suggested region.

THE PROBLEM

There already have papers deal with the problem of weakness of classical confidence regions. Formally, we quote one section from the book of Kendall, M and Stuart, A [1] to see the problem. "Simultaneous confidence intervals for several parameters cases fairly frequently arise in which we wish to estimate more than one parameter of a population, for example the mean and variance. The extension of the theory of confidence intervals for one parameter to this case of two or more parameters is a matter of very considerable difficulty. What we should like to be able to do given, say, two parameters θ_1 and θ_2 and two statistics t and u, is to make simultaneous interval assertions of the type

$$P(t_0 \le \theta_1 \le t_1 \text{ and } u_0 \le \theta_2 \le u_1) = 1-\alpha$$

This, however, is rarely possible. Sometimes we can make a statement giving a confidence region for the two parameters together, e.g. such as $P(w_0 \le \theta_1^2 + \theta_2^2 \le w_1) = 1 - \alpha$. But this is not entirely satisfactory; we do not know, so to speak, how much of the uncertainty of the region to assign to each parameter. It may be that, unless we are prepared to lay down some new rule on this point, the problem of locating the parameters in separate intervals is insoluble. Even for large samples the problems are severe. We may then find that we can determine intervals of the type

$$P(t_0(\theta_2) \le \theta_1 \le t_1(\theta_2)) = 1 - \alpha$$

and substitute a large sample estimate of θ_2 in the limits $t_0(\theta_2)$ and $t_1(\theta_2)$. This is very like the familiar procedure in the theory of standard errors, where we replace parameters occurring in the error variances by estimates obtained from the sample."

SURFACE GEOMETRY

From the 'Dupin's indicatrix' quadratic equation we can obtain the original definition of Gaussian Curvature as the ratio of the coefficient of second and first fundamental form. In 1826, Gauss has shown that Gaussian Curvature can be simplified and express as the coefficient of first fundamental form and it's derivative only. In 1886, R. Baltzer uses very complicated, tedious algebra prove this results. We now only layout his results here to be used later.

$$K = \frac{eg - f^2}{EG - F^2} = \frac{1}{(EG - F^2)^2}[(X_{uu} X_u X_v)(X_{vv} X_u X_v) - (X_{uv} X_u X_v)^2]$$

$$= \frac{1}{(EG-F^2)^2} \left| \begin{matrix} -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} & \frac{1}{2}E_u & F_u - \frac{1}{2}E_v \\ F_v - \frac{1}{2}G_u & E & F \\ \frac{1}{2}G_v & F & G \end{matrix} \right| - \left| \begin{matrix} 0 & \frac{1}{2}E_v & \frac{1}{2}G_u \\ \frac{1}{2}E_v & E & F \\ \frac{1}{2}G_u & F & G \end{matrix} \right| \quad (3.1)$$

The above form of Gaussian Curvature has two most useful points to us. Since it avoid on computing the coefficient of second fundamental form so it is much easier to compute. If we view this surface as a piece of not wet paper and do not involve stretching, shrinking, or tearing then the Gaussian Curvature has a so called bending invariant property. This bending leaves the distance between two points on the surface, measured along a curve on the surface, unchanged, and also the angle of two tangent obtain arbitrary direction at a point. When the curvilinear coordinate lines retain their position on the surface during the bending and the measurement of the coordinates remains the same then the coefficient of the first fundamental form and all their derivative with respective to the curvilinear coordinates also remain the same. A function containing E,F,G and their derivatives is therefore a bending invariant. This invariant property can also contain arbitrary function (u,v) or their derivatives of the functions. Properties of surfaces expressible by bending invariants are called intrinsic properties. It is always possible to reduce ds^2 of a surface to geodesic coordinate of the form

$$ds^2 = du^2 + G(u,v)\,dv^2 \quad (3.2).$$

The parameter u now measures the arc length along the geodesic lines v= constant, starting with u=0 on curve C, equation (3.2) is called the geodesic form of the line element and (u,v) form a set of geodesic coordinate; or, for short, geodesic set. On every surface we can find an infinite number of geodesic sets, depending on an arbitrary curve C along which the curve v

= constant can still be spaced in an arbitrary way. From the arc element above, we know in this case that E=1,F=0. We substitute these values into (3.1) and get Gaussian Curvature as:

$$G^2K = \frac{1}{4}G_u^2 - \frac{1}{2}GG_{uu} \text{ or}$$

equivalently

$$\mathrm{K} = -\frac{1}{\sqrt{\mathrm{G}}}\frac{\partial^2\sqrt{G}}{\partial u^2}$$

It is a well-known fact that the sum of interior angle of a plane triangle equal π. This fact has been generalized by Gauss to the arbitrary surface as follow.

Theorem 1: On a smooth surface with three geodesic lines constitute a triangle. Assume the three interior angles are

$\angle A, \angle B, \angle C$, then

$$\iint KdS = \angle A + \angle B + \angle C - \pi$$

Where dS represents the area element of the surface, K is Gaussian Curvature and double integral is triangle area bounded by three geodesic lines. We leave some explanations of detail in appendix. Only keep the main result as follow:

$$\iint KdS = -\int_0^{\angle A}\int_0^{u}\frac{\partial^2\sqrt{G(u,v)}}{\partial u^2}dudv = -\int_0^{\angle A}\frac{\partial\sqrt{G(u,v)}}{\partial u}\Big]_0^u dv$$

$$= \int_0^{\angle A}(1-\frac{\partial\sqrt{G}}{\partial u})dv = \int_0^{\angle A}(1+\frac{d\alpha}{dv})dv$$

$$= \angle \mathrm{A} + \alpha(\mathrm{u,v})]_0^{\angle A} = \angle A + \angle C - (\pi - \angle B)$$

$$= \angle A + \angle B + \angle C - \pi$$

we apply the fact that $\mathrm{dS} = \sqrt{\mathrm{EG - F^2}}\,dudv$

From above formula we can summarize that result as follows.

When $K = \frac{1}{a^2} > 0$ is a positive constant then the area of triangle bounded by three geodesic lines equal

$$\Delta = a^2(\angle A + \angle B + \angle C - \pi) > 0.$$

We conclude that the Gaussian curvature of a surface is a positive constant then the sum of three interior angles is greater than π. On the contrary, when $K = \frac{1}{a^2} < 0$ is a negative constant then the sum of three interior angles is less than π. It is a commonly known fact that when $K = 0$ the sum of three interior angles equal π. This result will apply to our confidence region. The following theorem was first published by Bonnet O. in the Journ. Ecole Polytechnique 19, 1848, pp1-146, as a generalization of Gauss' theorem on a geodesic triangle. Struik, D.J. [2] or Gray, A. [3] had similar proved theorem. Let us first take a smooth curve C, as the boundary of region R on which there are no points where the slope has discontinuities. Then we can contract C continuously without changing $\int_C d\theta$, since this is an integral multiple of 2π. Let A be a simply connected region, that is, a region which by continuous contraction of C can be reduced to a point. When C is reduced to approach a point then $\int_C d\theta = 2\pi$, and we have found the theorem that $\int_C \kappa_g ds + \iint_A KdA = 2\pi$. When

C consists of k arcs of smooth curves making exterior angles $\theta_1, \theta_2, \ldots \theta_k$ at the vertices $A_1, A_2 \ldots, A_k$ where the arcs meet, then we must keep in mind that $d\theta$ measures only the change of θ along the smooth arcs, where we measure $\int \kappa_g ds$, and not the jumps at the vertices. The total change in θ along C is still 2π, but only part of it is due to the change of θ along the arcs. The remainder is due to the angle $\theta_1, \theta_2, \ldots \theta_k$. For the sum of the line and area integral we therefore get $2\pi - \theta_1 - \theta_2 - \ldots - \theta_k$. This result is expressed in the Gauss-Bonnet theorem as follows:

Theorem 2: If the Gaussian curvature K of a surface is continuous in a simply connected region R bounded by a closed curve C composed of k smooth arcs making at the vertices exterior angles
$\theta_1, \theta_2, \ldots \theta_k$ then:

$$\int_C \kappa_g ds + \iint_A KdA = 2\pi - \sum \theta_i, \quad i=1,2,\ldots..k$$

Where κ_g represents the geodesic curvature of the arcs.

Example

The eight men received a certain drug. The recorded changes in systolic and diastolic blood pressure are listed below. The hypothesis to be tested is $H_0 \; \mu = 0$ vs $H_a \; \mu \neq 0$

Table 1. Data recorded changes in systolic and diastolic Blood pressure

Blood Pressure	Subject								Sample mean
	1	2	3	4	5	6	7	8	
systolic	-8	7	-2	0	-2	0	-2	1	-0.750
diastolic	-1	6	4	2	5	3	4	2	3.125

number of observation n=8 degree of freedom f=8-1=7

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} -0.750 \\ 3.125 \end{pmatrix} \qquad S = \begin{pmatrix} 17.357 & 6.393 \\ 6.393 & 4.696 \end{pmatrix}$$

Simultaneous ellipsoidal 95% confidence regions for the population mean μ_1 and μ_2 are given by the set of all (μ_1, μ_2) that satisfy the relationship

$$\langle(-0.75 - \mu_1), (3.125 - \mu_2)\rangle \begin{pmatrix} 0.116 & -0.157 \\ -0.157 & 0.427 \end{pmatrix} \begin{pmatrix} (-0.75 - \mu_1) \\ (3.125 - \mu_2) \end{pmatrix}$$

$$\leq \frac{7x2}{6x8} F_{2,6,0.05} = 0.292x5.1433 = 1.50$$

$$where \; T_\alpha^2 = 12.00$$

equivalenty,

$$(\mu_1 + 0.75)^2 - 2.707(\mu_1 + 0.75)(\mu_2 - 3.125) + 3.681(\mu_2 - 3.125)^2 \leq 12.931$$

Solving μ_1 and μ_2 we find the 95% confidence region for μ_1 and μ_2 to be the set of points (μ_1, μ_2) inside the region bounded by the curves

$$\mu_1 = (1.354\mu_2 - 4.980) \pm (11.556\mu_2 - 5.126 - 1.849\mu_2^2)^{\frac{1}{2}}$$

Roy's confidence intervals for μ_1 and μ_2 are obtained from appendix by choosing a=(1,0) and a=(0,1) respectively. In this example, these two confidence intervals are given by

$$-0.750 \pm (12x17.357/8)^{\frac{1}{2}} = -0.750 \pm 5.102$$

and

$$3.125 \pm (12x4.696/8)^{\frac{1}{2}} = 3.125 \pm 2.654$$

Equivalently

μ_1 bounded by (-5.852,4.352)

and

μ_2 bounded by (0.471,5.779)

Hence the area bounded by lines of μ_1 and μ_2 equal

A = 10.204x5.308 = 54.163

At this moment we don't know the Gaussian Curvature of $Hotelling-T^2$ distribution. For convenience of computation we may assume K equal some constant, say 1 and four interior angles are the same. Also let the four boundaries lines are geodesic lines then our Bonnet O. Theorem turns to be

$$KA = 2\pi - 4\vartheta$$

If we want area equal 54.163then $\vartheta = 76.46$

However, if we interest to use Gauss triangle theorem to fit this area then we may choose interior angle

$$KA = 3\vartheta - \pi$$

If we want area equal 54.163then $\vartheta = 78.054$

Of course, for the other selected K values or unequal angles it needs only some arithmetic adjustment.

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Appendix

The following results have been applied in this paper.

$$Claim1: \frac{\partial\sqrt{\mathrm{G}}}{\partial \mathrm{u}}\}_{u=0} = 1$$

Since at the point say A, curvature line u has reduced to a point and can not define the direction so we have G(0,v) = 0. If we take two near by points say, $M_0(u,0)$ and $M_1(u,v_1)$, let the arc length M_0M_1 approach to 0 then as $u \to 0$

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we can derive that $u \to v_1$

$$v_1 = \lim_{u\to 0}\frac{\int_0^{v_1}\sqrt{G}dv}{u} = \int_0^{v_1}\lim_{u\to 0}\frac{\sqrt{G(u,v)}-\sqrt{G(0,v)}}{u}dv = \int_0^{v_1}\frac{\partial\sqrt{G}}{\partial u}\}_{u=0}dv$$

Claim 2; If we let C be a geodesic line on the surface then any line of curvature u intersect C with angle $\alpha(\mathrm{u,v})$ we can derive

$$\frac{d\alpha}{dv} = -\frac{\partial\sqrt{G}}{\partial u}$$

Pr*oof* : See reference.

In the multivariate case $\mu \in R^p$, a p - dimentional space. Thus, an $(1-\alpha)\%$ confidence region will be an ellipsoidal region given by confidence intervals :

$$\mu \text{ vector: } T^2 = n(\bar{x}-\mu)' S^{-1}(\bar{x}-\mu) \le \frac{fp}{f-p+1}F_{p,\, f-p+1,\alpha}$$

The simultaneous confidence intervals for linear combination $a'\mu$ of μ are given by

$$a'\bar{x} - n^{\frac{-1}{2}}T_{\alpha}(a'Sa)^{\frac{1}{2}} \le a'\mu \le a'\bar{x} + n^{\frac{-1}{2}}T_{\alpha}(a'Sa)^{\frac{1}{2}}$$