ON ADOMIAN DECOMPOSITION METHOD FOR SOLVING GENERAL WAVE EQUATIONS ON TRANSMISSION LINES

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ABSTRACT

Adomian decomposition method is an efficient numerical scheme that provides a very good potential for the solution of physical applications that are modelled by non-linear differential equations. The method takes the form of a convergent series with easily computable components. In this paper, we applied the method to solve the general wave equations on transmission lines.

Keywords: Adomian Decomposition Method, Wave equations, Transmission lines, Adomian Polynomials.

INTRODUCTION

Adomian decomposition method is a relatively new numerical approach suitable for obtaining approximate solutions to differential equations. This method provides a direct scheme for solving differential equations without the need for linearization, perturbation, massive computation or any transformation, Makinde (2007b).

Adomian decomposition method had been applied by a lot of researchers to solve many mathematical problems in science and engineering. Makinde (2007a) used Adomian decomposition method in solving ratio-dependent predator–prey system with constant effort harvesting while Ibijola and Adegboyegun (2012) compared the Adomian decomposition method with Picard iteration method in the solution of non-linear differential equations. Biazar and Ebrahimi (2007) considered an approximation to the solution of telegraph equation by Adomian decomposition method while Abboui and Cherruault (1995) looked at some new ideas for proving convergence of decomposition methods. Ibijola and Adegboyegun (2008) used the Adomian decomposition method for the numerical solution of second-order ordinary differential equations while Makinde (2007b) applied the Adomian decomposition approach to solve an SIR epidemic model with constant vaccination strategy. Daftardar-Gejji and Jafari (2005) used Adomian decomposition method as a tool for solving a system of fractional differential equation. In this paper, the Adomian decomposition method is applied to solve general wave equations on transmission lines.

MATERIALS AND METHODS

Let us considered the equivalence circuit of a transmission line which is an infinitesimal piece of telegraph wire having length Δx with resistance $R\Delta x$, capacitance $C\Delta x$, inductance $L\Delta x$ and conductance $G\Delta x$ as shown in figure 1 below.

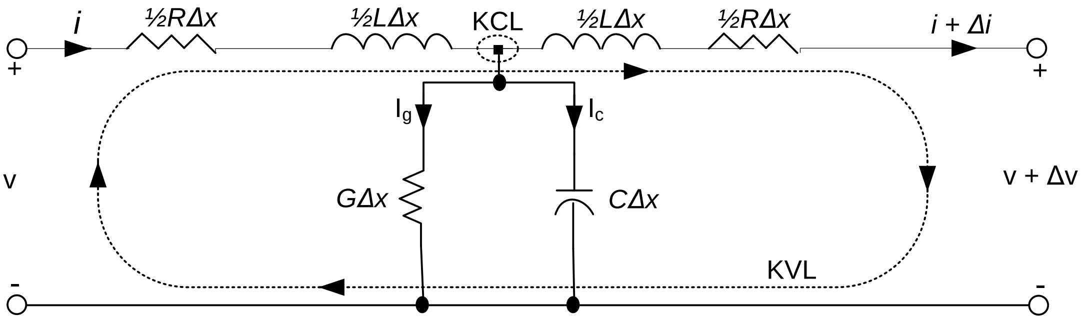

Figure 1: Equivalent Circuit of a Transmission Line

We are interested in determining the extent to which the output voltage and current differs from their input values as the length of the transmission line approaches a very small value. Therefore, applying the Kirchhoff's voltage and current laws to the symmetrical network of figure 1 and simplifying as appropriate, we have

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + LG\frac{\partial v}{\partial t} + R\left[Gv + C\frac{\partial v}{\partial t}\right] \quad (1)$$

and

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + CR\frac{\partial i}{\partial t} + G\left[Ri + L\frac{\partial i}{\partial t}\right] \quad (2)$$

The two equations above represent the general wave equations on a lossy transmission line, Hayt and Buck (2006), Oke (2012) and Oke (2015).

Dividing equation (2) by LC, we have

$$\frac{\partial^2 i}{\partial t^2} + \left(\frac{G}{C} + \frac{R}{L}\right)\frac{\partial i}{\partial t} + \left(\frac{G}{C}.\frac{R}{L}\right)i = \frac{1}{CL}\frac{\partial^2 i}{\partial x^2} \quad (3)$$

Let $\lambda = \frac{G}{C}$, $\beta = \frac{R}{L}$, $\phi = \frac{1}{CL}$, so that equation (3) now becomes

$$\frac{\partial^2 i}{\partial t^2} + (\lambda + \beta)\frac{\partial i}{\partial t} + \big((\lambda)(\beta)\big)i = \phi\frac{\partial^2 i}{\partial x^2} \quad (4)$$

Equation (4) will now be solved together with the initial conditions $i(x,0) = f(x)$ and $i_t(x,0) = g(x)$ by using Adomian decomposition method.

where i is the current through the conductor, $f(x)$ is the initial current and $g(x)$ is the initial speed of the current, Oke (2012) and Oke (2015).

To solve by the Adomian decomposition method, we use the operator $L_{tt} = \frac{\delta^2}{\delta t^2}$ and the inverse operator of L_{tt} given by $L_{tt}^{-1} = \int_0^t \int_0^t (.)dtdt$, Ibijola and Adegboyegun (2008).

Applying the inverse operator to both sides of (4), we have

$$i(x,t) = i(x,0) + \frac{\partial i(x,0)}{\partial t} + \int_0^t \int_0^t \left(\phi\frac{\partial^2 i}{\partial x^2} - (\lambda + \beta)\frac{\partial i}{\partial t} - \big((\lambda)(\beta)\big)i\right) dtdt \quad (5)$$

Putting the initial conditions in (5), we have

$$i(x,t) = f(x) + g(x)t + \int_0^t \int_0^t \left(\phi \frac{\partial^2 i}{\partial x^2} - (\lambda+\beta)\frac{\partial i}{\partial t} - \big((\lambda)(\beta)\big)i\right) dtdt \quad (6)$$

The Adomian decomposition method introduces the solution $i(x,t)$ as an infinite series of components of the form given by

$$i(x,t) = \sum_{n=0}^{\infty} i_n(x,t) \quad (7)$$

The components $i_n(x,t)$ of the solution $i(x,t)$will be determined recursively and the integrand on the right hand side of (6) will be expressed as the sum of an infinite series of polynomials as

$$\phi \frac{\partial^2 i}{\partial x^2} - (\lambda+\beta)\frac{\partial i}{\partial t} - \big((\lambda)(\beta)\big)i = \sum_{n=0}^{\infty} A_n \quad (8)$$

where $A_n(i_0, i_1, i_2, \ldots\ldots i_n)$ are the Adomian polynomials and should be computed. In general, we have

$$A_n(i_0, i_1, i_2, \ldots\ldots i_n) = \phi \frac{\partial^2 i_n}{\partial x^2} - (\lambda+\beta)\frac{\partial i_n}{\partial t} - \big((\lambda)(\beta)\big)i_n \quad (9)$$

for $n = 0, 1, 2, 3, 4, \ldots\ldots$

Putting (7) and (9) in (6), we have

$$\sum_{n=0}^{\infty} i_n(x,t) = f(x) + g(x)t + \sum_{n=0}^{\infty} \int_0^t \int_0^t \left(\phi \frac{\partial^2 i_n}{\partial x^2} - (\lambda+\beta)\frac{\partial i_n}{\partial t} - \big((\lambda)(\beta)\big)i_n\right) dtdt \quad (10)$$

The next step is to determine the components $i_n(x,t)$ for $n \geq 0$. The component $i_0(x,t)$ is first identified by all terms that arise from the initial conditions. That is

$$i_0(x,t) = i(x,0) + i_t(x,0)x = f(x) + g(x)x \quad (11)$$

The remaining components are determined by using the preceding components. Each term of the infinite series in (7) is given by the recurrent relation

$$i_{n+1}(x,t) = L_{tt}^{-1} A_n \quad (12)$$

That is

$$i_{n+1}(x,t) = \int_0^t \int_0^t \left(\phi \frac{\partial^2 i_n}{\partial x^2} - (\lambda+\beta)\frac{\partial i_n}{\partial t} - \big((\lambda)(\beta)\big)i_n\right) dtdt \quad (13)$$

for $n \geq 0$.

The first term of the series is given by

$$i_1(x,t) = \left(\sum_{j=0}^{1} \binom{1}{j} \phi^j (-\alpha\beta)^{1-j} g^{2j}(x)\right)\frac{t^3}{3!} - (\alpha+\beta)g(x)\frac{t^2}{2!} + \left(\sum_{j=0}^{1} \binom{1}{j} \phi^j (-\alpha\beta)^{1-j} f^{2j}(x)\right)\frac{t^2}{2!} \quad (14)$$

The second term of the series is given by

$$i_2(x,t) = \left(\sum_{j=0}^{2} \binom{2}{j} \phi^j (-\alpha\beta)^{2-j} g^{2j}(x)\right)\frac{t^5}{5!} - 2(\alpha+\beta)\left(\sum_{j=0}^{1} \binom{1}{j} \phi^j (-\alpha\beta)^{1-j} g^{2j}(x)\right)\frac{t^4}{4!} + (\alpha+\beta)^2 g(x)\frac{t^3}{3!} \left(+\sum_{j=0}^{2} \binom{2}{j} \phi^j (-\alpha\beta)^{2-j} f^{2j}(x)\right)\frac{t^4}{4!} - (\alpha+\beta)\left(\sum_{j=0}^{1} \binom{1}{j} \phi^j (-\alpha\beta)^{1-j} f^{2j}(x)\right)\frac{t^3}{3!} \quad (15)$$

The third term of the series is given by

$i_3(x,t) = \left(\sum_{j=0}^{3}\binom{3}{j}\phi^j(-\alpha\beta)^{3-j}g^{2j}(x)\right)\frac{t^7}{7!}$
$-3(\alpha+\beta)\left(\sum_{j=0}^{2}\binom{2}{j}\phi^j(-\alpha\beta)^{2-j}g^{2j}(x)\right)\frac{t^6}{6!}$
$+3(\alpha+\beta)^2\left(\sum_{j=0}^{1}\binom{1}{j}\phi^j(-\alpha\beta)^{1-j}g^{2j}(x)\right)\frac{t^5}{5!}$
$-(\alpha+\beta)^3 g(x)\frac{t^4}{4!} + \left(\sum_{j=0}^{3}\binom{3}{j}\phi^j(-\alpha\beta)^{3-j}f^{2j}(x)\right)\frac{t^6}{6!}$
$-2(\alpha+\beta)\left(\sum_{j=0}^{2}\binom{2}{j}\phi^j(-\alpha\beta)^{2-j}f^{2j}(x)\right)\frac{t^5}{5!}$
$+(\alpha+\beta)^2\left(\sum_{j=0}^{1}\binom{1}{j}\phi^j(-\alpha\beta)^{1-j}f^{2j}(x)\right)\frac{t^4}{4!}$ (16)

Continuing in this manner, we have the nth term of the series as

$i_n(x,t) = \sum_{k=1}^{n+1}((-1)^{n-k+1}\binom{n}{k-1}(\alpha+\beta)^{n-k+1}\left(\sum_{j=0}^{k-1}\binom{k-1}{j}\phi^j(-\alpha\beta)^{k-j-1}g^{2j}(x)\right)\frac{t^{(n+k)}}{(n+k)!}$
$+\sum_{k=1}^{n}((-1)^{n-k}\binom{n-1}{k-1}(\alpha+\beta)^{n-k}\left(\sum_{j=0}^{k}\binom{k}{j}\phi^j(-\alpha\beta)^{k-j}f^{2j}(x)\right)\frac{t^{(n+k)}}{(n+k)!}$ (17)

for n = 0, 1, 2, 3, 4, ...

Finally, the general solution is given by

$i(x,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{n+1}((-1)^{n-k+1}\binom{n}{k-1}(\alpha+\beta)^{n-k+1}\left(\sum_{j=0}^{k-1}\binom{k-1}{j}\phi^j(-\alpha\beta)^{k-j-1}g^{2j}(x)\right)\frac{t^{(n+k)}}{(n+k)!}$
$+\sum_{n=0}^{\infty}\sum_{k=1}^{n}((-1)^{n-k}\binom{n-1}{k-1}(\alpha+\beta)^{n-k}\left(\sum_{j=0}^{k}\binom{k}{j}\phi^j(-\alpha\beta)^{k-j}f^{2j}(x)\right)\frac{t^{(n+k)}}{(n+k)!}$ (18)

It is not possible to calculate all the term of the infinite series in (7). Therefore the exact solution of (4) with the initial conditions can not be entirely determined by this method. The final solution will therefore be approximated by series of the form

$\varphi_n(x,t) = \sum_{k=0}^{n-1} i_k(x,t)$ (19)

with the condition that

$\lim_{n\to\infty}\varphi_n(x,t) = i(x,t)$ (20)

RESULTS

Example 1

Consider the general wave equation

$\frac{\partial^2 i}{\partial t^2} + 3\frac{\partial i}{\partial t} + 2i = 4\frac{\partial^2 i}{\partial x^2}$

with the initial conditions $i(x,0) = Sin\ x,\ i_t(x,0) = 0$.

Applying the Adomian decomposition method, we have

$i_0 = Sin\ x$

$i_n = (-1)^n Sin\ x\ \sum_{j=1}^{n}(\binom{n-1}{j-1}(3)^{n-j}(6)^j\frac{t^{n+j}}{(n+j)!})$.

That is

$i_1 = (-1)^1 Sin\ x\binom{0}{0}(3)^0(6)^1\frac{t^2}{2} = -Sin\ x(3t^2)$

$i_2 = (-1)^2 Sin\ x[\binom{1}{0}(3)^1(6)^1\frac{t^3}{6} + \binom{1}{1}(3)^0(6)^2\frac{t^4}{24}]$

$= Sin\ x[3t^3 + \frac{3t^4}{2}]$

$i_3 = (-1)^3 Sin\, x[\binom{2}{0}(3)^2(6)^1\frac{t^4}{24} + \binom{2}{1}(3)^1(6)^2\frac{t^5}{120} + \binom{2}{2}(3)^0(6)^3\frac{t^6}{720}]$
$= -Sin\, x[\frac{9t^4}{4} + \frac{9t^5}{5} + \frac{3t^6}{10}]$
$i_4 =$
$(-1)^4 Sin\, x[\binom{3}{0}(3)^3(6)^1\frac{t^5}{120} + \binom{3}{1}(3)^2(6)^2\frac{t^6}{720} + \binom{3}{2}(3)^1(6)^3\frac{t^7}{5040} + \binom{3}{3}(3)^0(6)^4\frac{t^8}{40320}]$
$= Sin\, x[\frac{27t^5}{20} + \frac{27t^6}{20} + \frac{27t^7}{70} + \frac{9t^8}{280}]$
$i_5 = (-1)^5 Sin\, x[\binom{4}{0}(3)^4(6)^1\frac{t^6}{720} + \binom{4}{1}(3)^3(6)^2\frac{t^7}{5040}$
$+\binom{4}{2}(3)^2(6)^3\frac{t^8}{40320} + \binom{4}{3}(3)^1(6)^4\frac{t^9}{362880} + \binom{4}{4}(3)^0(6)^5\frac{t^{10}}{3628800}]$
$= -Sin\, x[\frac{27t^6}{40} + \frac{27t^7}{35} + \frac{81t^8}{280} + \frac{t^9}{140} + \frac{3t^{10}}{1400}]$
The five terms approximation to the solution is
$i(x,t) = Sin\, x[1 - 3t^2 + 3t^3 - \frac{3t^4}{4} - \frac{9t^5}{20} + \frac{15t^6}{40} - \frac{27t^7}{70} - \frac{72t^8}{280} - \frac{t^9}{140} - \frac{3t^{10}}{1400}]$

Example 2

Consider the general wave equation
$\frac{\partial^2 i}{\partial t^2} + 2\frac{\partial i}{\partial t} + i = \frac{\partial^2 i}{\partial x^2}$
with the initial conditions $i(x,0) = Cos\, x,\ i_t(x,0) = 0.$
Applying the Adomian decomposition method, we have
$i_0 = Cos\, x$
$i_n = (-1)^n Cos\, x\ \sum_{j=1}^{n}(\binom{n-1}{j-1}(2)^{n-j}(2)^j\frac{t^{n+j}}{(n+j)!}).$
That is
$i_1 = (-1)^1 Cos\, x\ \binom{0}{0}(2)^0(2)^1\frac{t^2}{2} = -Cos\, x(t^2)$
$i_2 = (-1)^2 Cos\, x[\binom{1}{0}(2)^1(2)^1\frac{t^3}{6} + \binom{1}{1}(2)^0(2)^2\frac{t^4}{24}]$
$= Cos\, x[\frac{2t^3}{3} + \frac{t^4}{6}]$
$i_3 = (-1)^3 Cos\, x[\binom{2}{0}(2)^2(2)^1\frac{t^4}{24} + \binom{2}{1}(2)^1(2)^2\frac{t^5}{120} + \binom{2}{2}(2)^0(2)^3\frac{t^6}{720}]$
$= -Cos\, x[\frac{t^4}{3} + \frac{2t^5}{15} + \frac{t^6}{90}]$
$i_4 =$
$(-1)^4 Cos\, x[\binom{3}{0}(2)^3(2)^1\frac{t^5}{120} + \binom{3}{1}(2)^2(2)^2\frac{t^6}{720} + \binom{3}{2}(2)^1(2)^3\frac{t^7}{5040} + \binom{3}{3}(2)^0(2)^4\frac{t^8}{40320}]$
$= Cos\, x[\frac{2t^5}{15} + \frac{t^6}{15} + \frac{t^7}{105} + \frac{t^8}{2520}]$
$i_5 = (-1)^5 Cos\, x[\binom{4}{0}(2)^4(2)^1\frac{t^6}{720} + \binom{4}{1}(2)^3(2)^2\frac{t^7}{5040}$
$+\binom{4}{2}(2)^2(2)^3\frac{t^8}{40320} + \binom{4}{3}(2)^1(2)^4\frac{t^9}{362880} + \binom{4}{4}(2)^0(2)^5\frac{t^{10}}{3628800}]$
$= -Cos\, x[\frac{2t^6}{45} + \frac{8t^7}{315} + \frac{t^8}{210} + \frac{t^9}{2835} + \frac{t^{10}}{113400}]$
The five terms approximation to the solution is
$i(x,t) = Cos\, x[1 - t^2 + \frac{2t^3}{3} - \frac{t^4}{6} + \frac{t^6}{90} - \frac{t^7}{63} - \frac{11t^8}{2520} - \frac{t^9}{2835} - \frac{t^{10}}{113400}]$

Example 3

Consider the general wave equation

$$\frac{\partial^2 i}{\partial t^2}+5\frac{\partial i}{\partial t}+6i=16\frac{\partial^2 i}{\partial x^2}$$

with the initial conditions $i(x,0)=e^x,\ i_t(x,0)=x.$

Applying the Adomian decomposition method, we have

$$i_0=e^x+xt$$

$$i_n=(-1)^n e^x\sum_{k=1}^{n+1}\binom{n}{k-1}(5)^{n-k+1}(22)^{k-1}\frac{t^{(n+k)}}{(n+k)!}$$

$$+(-1)^n x\sum_{k=1}^{n}\binom{n-1}{k-1}(5)^{n-k}(22)^{k}\frac{t^{(n+k)}}{(n+k)!}$$

The solution to this problem is given by

$$i(x,t)=e^x\left[\sum_{n=0}^{\infty}\sum_{k=1}^{n+1}(-1)^n\binom{n}{k-1}(5)^{n-k+1}(22)^{k-1}\frac{t^{(n+k)}}{(n+k)!}\right]$$

$$+x\sum_{n=0}^{\infty}\sum_{k=1}^{n}(-1)^n\binom{n-1}{k-1}(5)^{n-k}(22)^{k}\frac{t^{(n+k)}}{(n+k)!}$$

Following the same procedure as in examples 1 and 2 above, we have the seven terms approximation to the solution of this problem as

$$i(x,t)=e^x[1+5t^2-\frac{25t^3}{3}+\frac{175t^4}{12}-\frac{75t^5}{4}+\frac{1475t^6}{72}$$
$$-\frac{1375t^7}{72}+\frac{375t^8}{32}-\frac{18125t^9}{4536}+\frac{1625t^{10}}{2268}-\frac{3125t^{11}}{49896}+\frac{625t^{12}}{299376}]$$
$$+x[t-\frac{5}{2}t^2+\frac{19t^3}{6}-\frac{65t^4}{24}+\frac{211t^5}{120}-\frac{133t^6}{144}$$
$$+\frac{2059t^7}{5040}+\frac{57t^8}{32}+\frac{879t^9}{1120}+\frac{47t^{10}}{336}+\frac{369t^{11}}{30800}+\frac{3t^{12}}{6160}+\frac{3t^{13}}{400400}]$$

CONCLUSION

Kirchhoff's voltage and current laws were applied to the symmetrical network which represents the equivalent circuit of a transmission line. This now resulted into non-linear second order hyperbolic partial differential equation which represents the general wave equations on transmission lines. The equation was now solved by the use of Adomian decomposition method. The mathematical computation of the resulting partial differential equation was simplified by the use of Maple 9 package.

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