ON THE EQUILIBRIUM POINTS OF THREE MUTUALLY COMPETING, SYMMETRIC AND CONTINUOUS TIME REPRODUCING ORGANISMS

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ABSTRACT

This work deals with the problem of three mutually competing species within a stable ecosystem. The model is represented by a system of non-linear ordinary differential equations. As much as six non-extinction equilibrium states have been obtained depending on the value of various interaction or efficiency parameters. A set of numerical schemes for the discrete solution of the resulting system have been developed using the technique of non-local approximation and renormalisation of the denominator function which are the bedrock of non-standard finite difference method. The new scheme confirms that the analytic equilibrium points of the system compares favourably with a Runge kutta scheme of order four.

Keywords: Mutually competing, Continuous time reproducing organism, Nonstandard Method, Equilibrium point, Non-local approximation.

INTRODUCTION

The model of a system of three competing populations in a single ecological environment can be represented by the following system of ordinary difference equations, Beltrami (1986).

$$\begin{aligned} \dot{X} &= X(1 - X - \propto Y - \beta Z) \\ \dot{Y} &= Y(1 - \beta X - Y - \alpha Z) \\ \dot{Z} &= Z(1 - \alpha X - \beta Y - Z) \end{aligned} \tag{1}$$

$X_i$is the population density of the i^{th} population,$\alpha\ and \beta$are the efficiency parameters. $\alpha,\ \beta$ represent the effect on the group of a specie due to its interactions with another group of specie. In the case or prey predator model $\alpha\ and\ \beta$may represent the rate at which the predator catches her prey and the growth rate of the predator resulting from her catches. The above model was proposed by Lotka-Voltera as explained by Beltrami (1986). In this model two crucial points are always considered as the key factors. They are: reproduction and extinction. All the assumptions of Beltrami (1986) are adopted. Such assumptions include fairly stable ecosystem, a conducive atmosphere for competition, continuous or discrete reproduction intervals and disregard to age or sex of any member.

The biological foundation of this problem is to obtain the equilibrium state of the eco-system. This represents the values for which the population densities of the three species remain constant for all time. In this case, there can be no further change in the numerical value of the population of any of the three species with respect to time.

In this work, we shall consider only the continuous time reproduction model. We shall investigate the analytic equilibrium point and use numerical experiment to investigate the existence of other non -extinction equilibrium points. The motivations for this work are the Malthusian theory and the earlier works of Mickens (1994).

LITERATURE REVIEW
Preliminaries

In this section, we will give some basic definitions and a theorem that will be useful for our derivations.
Let,

$$\frac{dy}{dt} = \underline{f}\left(t,\underline{y}\right) \underline{y}(to) = \underline{\gamma}$$
$$\underline{y} = (y_{1,}\ y_{2,}\ ..., y_n)^T \qquad (2)$$
$$\underline{f}\left(t,\underline{y}\right) = (f_{1,}\ f_{2,} ..., f_n)^T$$
$$\underline{\gamma} = (y_0, y_0, y_0^{n-1})^T$$

be the vector notation of n first order system of differential equation.
Definition 1: Any function φ, defined on an interval **I** and processing at least n derivatives that are continuous on **I**, which when substitutes into an nth order ordinary differential equation reduces the equation to an identity is said to be a solution of the equation on the interval **I**, Zill and Cullen (2005). i.e

$$f\left(x, \varphi(x), \varphi'(x), \ldots, \varphi^n(x)\right) = 0 \ \ \forall\, x \in I \qquad (3)$$

Definition 2: The interval **I** of the definition of solution of an IVP is that interval $\mathbf{I_0}$ in which the {solution} y(x) is defined, y(x) is differentiable and contains the initial point $\mathbf{I_0 C\ I}$, Zill and Cullen (2005).
Theorem: Picard's Theorem on the Uniqueness of Solution
If $\underline{f}(x, \underline{y})$ satisfies the following conditions:

(i) $\underline{f}(x , \underline{y})$ is a real-valued function defined and continues for $x \in (x_0, b)$, $\underline{y} \in R^n$.
(ii) $\underline{f}(x, \underline{y})$ satisfy Lipschitz conditions in the domain U of its definition where
$U = x_0 \leq t \leq x_N\}$ x $\{y_0 \leq y \leq y_N\}$

Then for every $\underline{y}_{0,}$ the initial value problem (2) has a unique solution $\underline{y}(x)$, $t \in (x_0, b)$.
Definition 3**:**The zeros of the function f in the equation (1.2) is a critical point. A point $\underline{y} \in R$ is called a fixed point or equilibrium point of the dynamical system defined by (1.2) if $f(\underline{y}) = 0$. If c is any critical point of fi.e f(c) = 0 then y(x) = c is a constant solution of the differential equation, Zill and Cullen (2005).
Definition 4: Equilibrium point

Any constant solution of the system (2) is called an equilibrium point of the system: thus any orbit along which the derivative $\underline{y}$ is identically zero is a "fixed point" and is referred to as an equilibrium point, Beltrami (1986).

Equilibrium Points of the System

We will assume that our system of equation (1) satisfies the Picard's theorem and hence that there exist such region for which unique solution exist. Thus:
The trivial equilibrium points of system (1) when $\overline{X}_i = 0$, i = 1, 2, 3 is (0,0,0)
The partial extinction equilibrium points are ((1,0,0), (0,1,0), (0,0,1)).
The analytic non-extinction equilibrium points can be obtained by solving the system:

$$\overline{X} = \begin{matrix} X + \alpha Y + \beta Z = 1 \\ \beta X + Y + \alpha Z = 1 \\ \alpha X + \beta Y + Z = 1 \end{matrix} \qquad (4)$$

With the following augmented matrix

$$\begin{bmatrix} 1 & \alpha & \beta & 1 \\ \beta & 1 & \alpha \vdots & 1 \\ \alpha & \beta & 1 & 1 \end{bmatrix}$$

CaseI: when $x_1 \neq 0$, $x_2 \neq 0$, $x_3 \neq 0$

The solution to the augmented above is

$$\overline{Z} = \frac{(\alpha^2-\beta)(1-\beta)+(1-\alpha)(1-\alpha\beta)}{(\alpha^2-\beta)(\alpha-\beta^2)+(1-\alpha\beta)^2}$$

$$\overline{Y} = \left(\frac{1-\beta}{1-\alpha\beta}\right) - \left(\frac{(\alpha-\beta^2)}{1-\alpha\beta)}\right) Z$$

$\overline{X} = 1-\alpha\overline{Y} - \beta\overline{Z}$; $\alpha\beta \neq 1$

Case II: If $\overline{X} = 0, \overline{Y} \neq 0, \overline{Z} \neq 0$

The resulting equation is as given:

$$X = 0$$
$$Y(1 - Y - \alpha Z) = 0$$
$$Z(1 - \beta Y - Z) = 0$$
$$\Rightarrow \quad Y + \alpha Z = 1$$
$$Z + \beta Y = 1$$

The solution is

$\overline{Z} = \frac{1-\beta}{1-\alpha\beta}$, $\overline{Y} = \frac{1-\alpha}{1-\alpha\beta}$, $\overline{X} = 0$

Case III

Similarly when $\overline{Y} = 0$, $\overline{X} \neq 0, \overline{Z} \neq 0$

We obtain $\overline{X} = (\frac{1-\beta}{1-\alpha\beta}, 0, \frac{1-\alpha}{1-\alpha\beta})$

Case IV

Similarly when $Z = 0$, $\overline{X} \neq 0$, $X_2 \neq 0$

We obtain $(\frac{1-\alpha}{1-\alpha\beta}, \frac{1-\beta}{1-\alpha\beta}, 0)$

Case V

Similarly when $Z = 0$, $X \neq 0$, $Y \neq 0$. We obtain $(0, \frac{1-\alpha}{1-\alpha\beta}, \frac{1-\beta}{1-\alpha\beta})$

From the analysis point of view the behaviour of the system clearly depends on the value of $\alpha and \beta$ and the relationship between the two.We will test for these equilibrium points numerically for various types of mutually related species. We will also examine the system for the following situations e.g

1. $\alpha = 0$, $\beta = 0$: when species are harmless from one type to another, competition is negligible.
2. $\alpha = 1$, $\beta = 1$:X can prey on Y, Y can prey on Z and Z can prey on$X$ e.g eagle, snake, man. Eagle can prey on snake, snake is harmful to man, man can prey on eagle etc.
3. $\alpha = 1$, $\beta = 0$, or $\alpha = 0$, $\beta = 1$: These are models which can be used for pest control competitive ability is one sided between any two specie.
4. $\alpha, \beta \in (0,1)$ i.e $\alpha\beta < 1$, e.g colony of speciesin co-operation e.g bacterial colony
5. $\alpha, \beta > 1$: This implies that the three species are strongly competing .

METHODOLOGY

Construction of Numerical Schemes

For the construction of the new numerical schemes we will use the rules 2 and 3 of the non-standard modeling rules as proposed by Mickens (1994).

Rule 2 of Non-Standard Modeling (Mickens 1994)

Denominator function for the discrete derivatives must be expressed in terms of more complicated function of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of h in the denominator. For instance, consider

$$\frac{dy}{dx} = y(1-y) \tag{5}$$

This is in form of a logistic equation.

If we consider the discrete model of the first order differential equation

$$y' = f(x, y) \tag{6}$$

in the form given by

$$y_{n+1} = y_n + \varphi f(x_n y_n) \quad \text{OR} \quad \frac{y_{n+1} - y_n}{\varphi\,(h)} = f(x_n y_n) \tag{7}$$

If instead of the conventional "h" the denominator functionsφ (h) is given by,

$$\varphi\,(h) = e^h - 1 \tag{8}$$

Then substituting equation (8) in equation (7) gives

$$\frac{y_{n+1} - y_n}{e^h - 1} = f(x_n y_n) = y_n(1 - y_n) \tag{9}$$

It must be stated here that the selection of an appropriate denominator is an 'art', Mickens (1994). Close examinations of differential equation, for which the exact schemes are known, shows that the denominator function generally are functions that are related to particular solutions or properties of general solution to the differential equation. This therefore places great importance on the necessity of the modeler to obtain as much analytic knowledge as possible about the solutions to the differential equation.

Rule 3 of Non-Standard Modeling (Mickens 1994)

The non-linear terms must in general be modeled (approximated) non-locally on the computational grid or lattice in many different ways, for instance, the non-linear terms $y^2$and y^3 can be modeled as follows,

$$y^2_n \approx y_n y_{n+1} \tag{10}$$

$$y^2 \approx \frac{y_{n+1} + y_n}{2} y_n \tag{11}$$

$$y^3 \approx y^2_n y_{n+1} \tag{12}$$

$$y^3 \approx y^2_n \frac{y_{n+1} + y_n}{2} \tag{13}$$

In general any linear combination of the expressions listed in(10) to (13) with the sum of the co-efficient equal to 1 approximates y^2 or y^3 the error being of order $O(h^2)$ for sufficiently smooth y (see Anguelov and Lubuma 2003). In this way the function *f* in equation (6) may be approximated by an expression which contains certain number of free parameters. The particular form selected from equations (7) to (10) depends on the full discrete model.

Derivation of the Non-Standard Numerical Scheme

Consider $\dot{X} = X\,(1 - X - \alpha Y - \beta Z)$

i.e $\quad = X - X^2 - \alpha XY - \beta XZ$ (14)

Applying rule 2 by selecting a suitable denominator function φ

$X_{k+1} = X_k + \varphi f(X_k, Y_k, Z_k)$

(15)

Applying non-local approximation to the non-linear terms

$X_{k+1} = X_k + \varphi(X_k - aX_{k+1}X_k - bX_k - \alpha X_k Y_k - \beta X_k)$

$= X_k + \varphi X_k$ - a$\varphi X_{k+1}X_k$ - bφX_k - $\alpha\varphi X_k Y_k$ - $\beta\varphi X_k Z_k$

$= X_k (1 + \varphi - b\varphi X_k$ - $\alpha\varphi Y_k$ - $\beta\varphi Z_k) - a\varphi X_{k+1}X_k$

(16)

$X_{k+1} [(1 + a\varphi X_k)] = X_k(1 + \varphi - b\varphi X_k - \alpha\varphi Y_k - \varphi Z_k)$

(17)

$X_{k+1} [(1 + a\varphi X_k)] = X_k(1 + \varphi) - \varphi X_k(b + \alpha Y_k + \beta Z_k)$

(18)

$X_{k+1} = \frac{X_k(1+\varphi) - \varphi X_k(b+\alpha Y_k+\beta Z_k)}{1+a\varphi X_k}$

(19)

Similarly we can obtain

$Y_{k+1} = \frac{Y_k\ (1+\varphi) - \varphi Y_k\ (b+\beta X_k+\alpha Z_k)}{1+a\varphi Y_k}$

(20)

$Z_{k+1} = \frac{Z_k(1+\varphi) - \varphi Z_k(b+\alpha X_k+\beta Y_k)}{1+a\varphi Z_k}$

(21)

The Non-standard numerical scheme is given by

$NZ_{K+1} = \frac{NZ_k(1+\varphi) - \varphi NZ_k(b+\alpha NX_k+\beta NY_k)}{(1+a\varphi NZ_k)}$

(22)

$NY_{k+1} = \frac{NY_k(1+\varphi) - \varphi NY_k(b+\beta NX_{jk}+\alpha NZ_k)}{(1+a\varphi NY_k)}$

(23)

$NX_{k+1} = \frac{NX_k(1+\varphi) - \varphi NX_k(b+\alpha NX_k(b+\alpha NY_k+\beta NZ_k)}{(1+a\varphi NX_k)}$ (24)

$a + b = 1,\ \varphi = \frac{(e^{\lambda h}-1)}{\lambda}$

Derivation of the Exact Numerical Scheme

From the analytic solution of Case I above, we can obtain the exact scheme

$Ez_{K+1} = \frac{(\alpha^2-\beta)(1-\beta)+(1-\alpha)(1-\alpha\beta)}{(\alpha^2-\beta)(\alpha-\beta^2)+(1-\alpha\beta)^2}$

$Ey_{K+1} = \left(\frac{1-\beta}{1-\alpha\beta}\right) - [\frac{(\alpha-\beta^2)}{1-\alpha\beta)}]z_K$

(25)

$EX_{K+1} = 1 - \alpha Y_K - \beta Z_K$

when $\alpha\beta \neq 1$

Derivation of Runge Kutta Scheme of Order 4

$RX_{K+1} = RX_K + \frac{h}{6}(P_1 + 4P_3 + P_4)$ (26)

$P_1 = f(t_k, RX_k) = RX_k(1 - RX_k - \propto RY_k - \beta RZ_k)$ (27)

$P_2 = f\left(t_k + \frac{h}{2}, RX_k + \frac{hP_1}{2}\right)$

$$= (RX_k + \frac{hP_1}{2})\left(1 - RX_k + \frac{hP_1}{2} - \propto RY_k - \beta RZ_k\right) \quad (28)$$

$$P_3 = f\left(t_k + \frac{h}{2}, [RX_k + \frac{h}{4}(P_1 + P_2)]\right)$$

$$= [RX_k + \frac{h}{4}(P_1 + P_2)]\left(1 - [RX_k + \frac{h}{4}(P_1 + P_2)] - \propto RY_k - \beta RZ_k\right) \quad (29)$$

$$P_4 = f(t_k + h, [RX_k - hP_2 + 2hP_3])$$

$$= [RX_k - hP_2 + 2hP_3](1 - [RX_k - hP_2 + 2hP_3] - \propto RY_k - \beta RZ_k) \quad (30)$$

$$RY_{K+1} = RY_K + \frac{h}{6}(W_1 + 4W_3 + W_4) \quad (31)$$

$$W_1 = f(t_k, RY_k) = RY_K(1 - \beta RX_k - RY_k - \alpha RZ_k) \quad (32)$$

$$W_2 = f\left(t_k + \frac{h}{2}, [RY_k + \frac{hW_1}{2}]\right)$$

$$= [RY_k + \frac{hW_1}{2}]\left(1 - \beta RX_k - [RY_k + \frac{hW_1}{2}] - \alpha RZ_k\right)$$

(33)

$$W_3 = f\left(t_k + \frac{h}{2}, [RY_k + \frac{h}{4}(W_1 + W_2)]\right)$$

$$= [RY_k + \frac{h}{4}(W_1 + W_2)]\left(1 - \beta RX_k - [RY_k + \frac{h}{4}(W_1 + W_2)] - \alpha RZ_k\right) \quad (34)$$

$$W_4 = f(t_k + h, [RY_k - hW_2 + 2hW_3])$$

$$= [RY_k - hW_2 + 2hW_3](1 - \beta RX_k - [RY_k - hW_2 + 2hW_3] - \alpha RZ_k)$$

(35)

$$RZ_{K+1} = RZ_K + \frac{h}{6}(M_1 + 4M_3 + M_4) \quad (36)$$

$$M_1 = f(t_k, Rz_k) = RZ_K(1 - \alpha RX_k - \beta RY_k - RZ_k) \quad (37)$$

$$M_2 = f\left(t_k + \frac{h}{2}, [Rz_k + \frac{hM_1}{2}]\right)$$

$$= [Rz_k + \frac{hM_1}{2}]\left(1 - \alpha RX_k - \beta RY_k - [Rz_k + \frac{hM_1}{2}]\right)$$

(38)

$$M_3 = f\left(t_k + \frac{h}{2}, [Rz_k + \frac{h}{4}(M_1 + M_2)]\right)$$

$$= [Rz_k + \frac{h}{4}(M_1 + M_2)]\left(1 - \alpha RX_k - \beta RY_k - [Rz_k + \frac{h}{4}(M_1 + M_2)]\right)$$

(39)

$$M_4 = f(t_k + h, [RZ_k - hM_2 + 2hM_3])$$

$$= [RZ_k - hM_2 + 2hM_3](1 - \alpha RX_k - \beta RY_k - [RZ_k - hM_2 + 2hM_3])$$

(40)

A suite of programs have been developed to test the algorithms of the schemes developed and the results obtained were analyzed and compared with the analytic properties of the original model.

RESULTS

The results of the numerical experiment on the schemes derived are presented in the following 3D graphs.

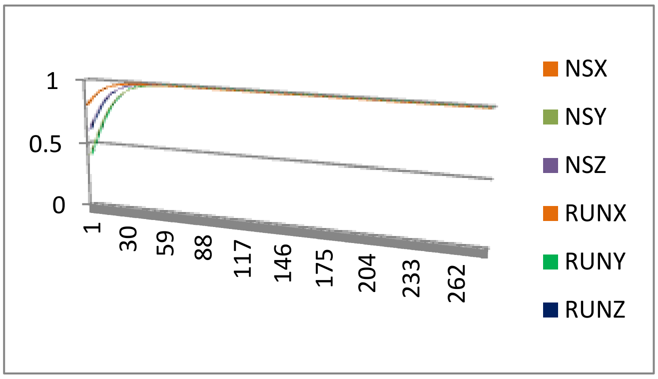

Figure1: Orbit of the schemes for $\propto = 0 , \beta = 0$ the equilibrium state is (1,1,1)
NOTE: "NSX" = Non-standard Scheme of X, "RUNX"= Runge Kutta Scheme of X

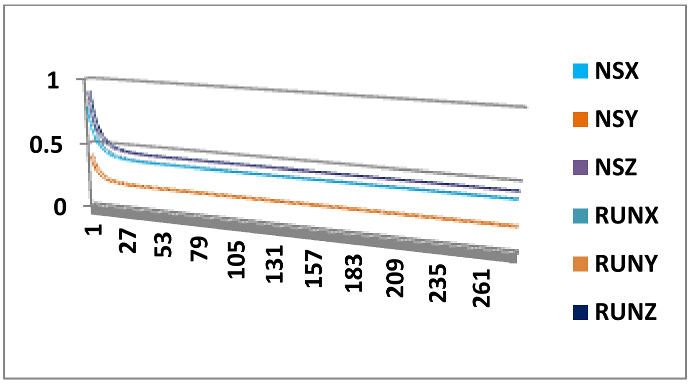

Figure 2: Orbit of the schemes $\propto= 1 , \beta = 1$
the equilibrium state is (0.38, 0.19,0.43) i.e (X , Y, Z) such that $X + Y + Z = 1$

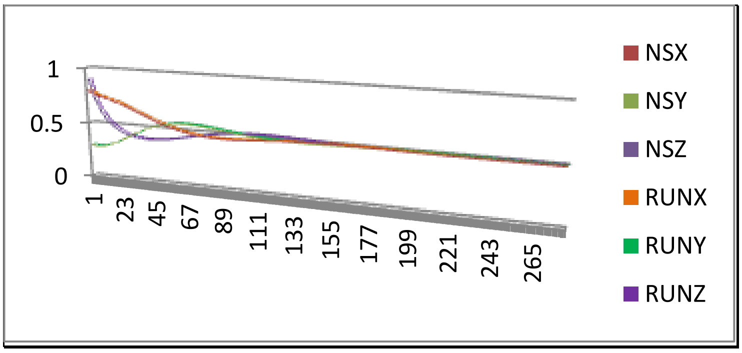

Figure 3: Orbit of the schemes for $\propto = 1 , \beta = 0$ or $\propto = 0 , \beta = 1$
the equilibrium state is (0.5 ,0.5, 0.5)

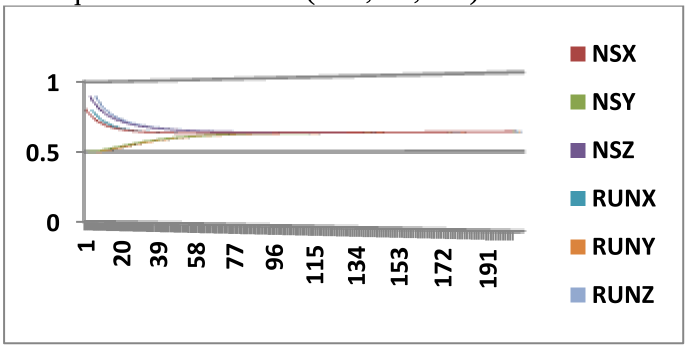

Figure 4: Orbit of the schemes for $\alpha, \beta \epsilon (0,1) and \alpha\beta < 1$
the equilibrium point is (0.625, 0.625, 0.625) which is (k, k, k) where k = $\frac{1}{1+\alpha+\beta}$.

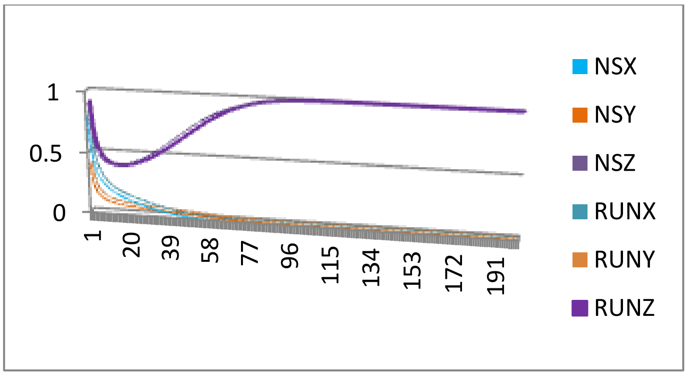

Figure 5: Orbit of the schemes for $\alpha, \beta > 1$ i.e $\alpha\beta > 1$
the equilibrium point is (0, 0, 1) for $\alpha = 3,\ \beta = 2$ which is (k, k, k) where k = $\frac{1}{1+\alpha+\beta}$
The equilibrium points are (1,0,0), (0,1,0) & (0,0,1) depending on the initial values.

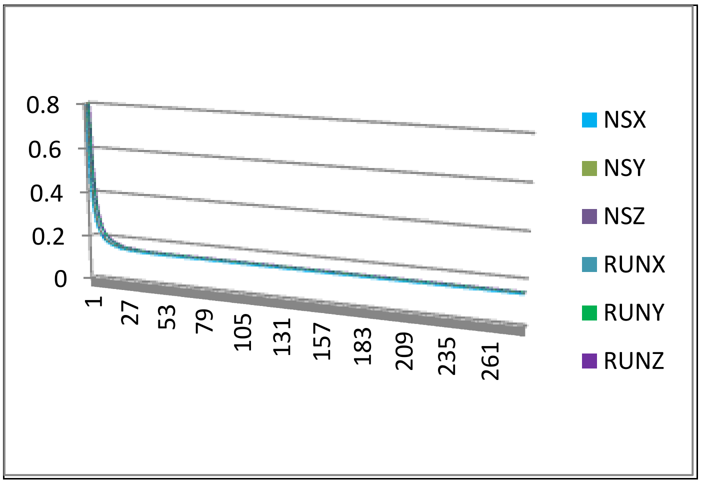

Figure 6: Orbit of the schemes for $\alpha, \beta > 1$ i.e $\alpha\beta > 1$
The equilibrium point is (0.14286, 0 .14286, 0.14286) ($\alpha = \beta = 3$) which is (k, k, k) where k = $\frac{1}{1+\alpha+\beta}$
The equilibrium points are (1,0,0), (0,1,0) & (0,0,1) depending on the initial values.

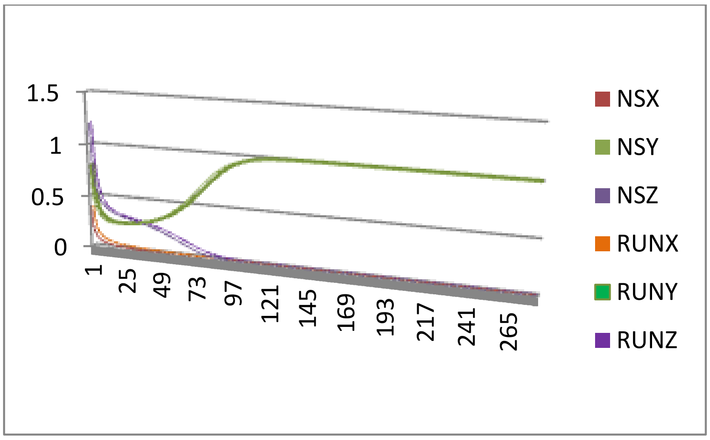

Figure 7: Orbit of the schemes for $\alpha, \beta > 1, \alpha \neq \beta$ i.e $\alpha\ \beta > 1$
the equilibrium point is (0, 1 ,0) for $\alpha = 2, \beta = 3$
Generally for $0 \neq \alpha\beta \neq 1$ the result confirm the following set of equilibrium points

i. (1,0,0), (0,1,0), (0,0,1),(0,0,0)
ii. (k, k, k) where k = $\frac{1}{1+\alpha+\beta}$

iii. $\left[0, \frac{1-\alpha}{1-\alpha\beta}, \frac{1-\beta}{1-\alpha\beta}\right], \left[\frac{1-\beta}{1-\alpha\beta}, 0, \frac{1-\alpha}{1-\alpha\beta}\right], \left[\frac{1-\alpha}{1-\alpha\beta}, \frac{1-\beta}{1-\alpha\beta}, 0\right]$

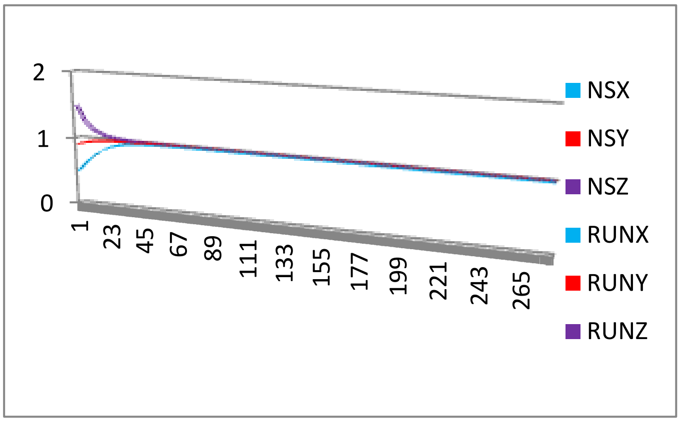

Figure 8: Orbit of the schemes for $\propto = 0, \beta = 0$
the equilibrium state is $(1, 1, 1)$ for any non-zero initial value

DISCUSSION

From the results of the numerical experiment, we have obtained six types of equilibrium points for this model. We also used this to confirm the analytic equilibrium points.

CONCLUSION

The use of non-standard method helped to reduce the amount time used in the construction of the schemes and the efficiency of the computer software in terms of work area storage and run time. The schemes produced orbits that preserve the dynamics of the original system.

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