European Journal of Physical and Agricultural Sciences
Vol. 1 No. 1, 2013

APPLICATIONS OF FIXED POINT IN MENELAUS THEOREM

Süheyla Elmas & Seyfullah Hızarcı
Department of Mathematics
Atatürk University Erzurum, Turkey
Corresponding Author Email: seyfullahhizarci@atauni.edu.tr

ABSTRACT

In this study, we showed that “the Menelaus theorem has two fixed points by using the homothety and dilation.

Keywords: Fixed point, Euclid geometry, Menelaus Theorem, Dilation and Homothety.

INTRODUCTİON

A study like this study was made by Stanley R. Clemen in 1973. According to Clemen(1973) this new way, the conversions from the classical synyhetic geometry to main appearance of modern mathematics is examined as the transfer theorems. The use of this two views simultaneously will be the practice of the theorems of fixed points. These theorems are wide collections spreading through the heart of mathematics. In this study, even if the fixed point theorem is limited, it will be moved on some theorems of Euclidean geometry.

In this study, differently from the Clemen’s study, by the homothety and dilation defined by director vector the tangent vector space, it is showed that the Menelaus theorem has two fixed points in a triangle.

DEFİNİTİON 1.1

Let X be nonempty set. Given a mapping $T: X \to X$, from a set X onto itself, a point x in X is called a fixed point for T if $Tx = x$. [2]

DEFİNİTİON 1.2

Let X be a nonempty set and let $T: X \to X$ be a mapping. In this case

$$ToT(x) = T(T(x))$$

is a mapping, from X onto itself, and this mapping is called second iteration of T. Generally $ToToTo \ldots oT(x)$ is a mapping and this mapping is called n. iteration of T. This mapping is represented like below,

$$ToToTo \ldots\ldots oT(x) = T(T(\ldots.T(x)\ldots\ldots)).$$ [2]

DEFİNİTİON 1.3

A tangent vector V_p to $R^{\mathbf{3}}$ consists of two points of R^3 which its vector part V and its point of application P. (See Figure 1.1) [3]

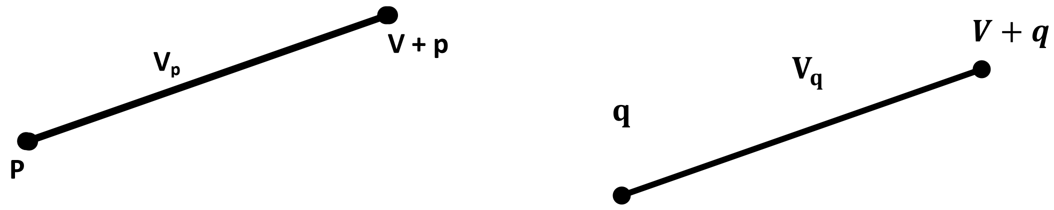

FİGURE 1.1 (TANGENT VECTOR)

DEFİNİTİON 1.4

Let P be a point of R^3. The *set* $T_p(R^3)$ *consisting* of all tangent vectors that have p as point of application is called the tangent space of R^3 at P . (See Figure 1. 2) [4]

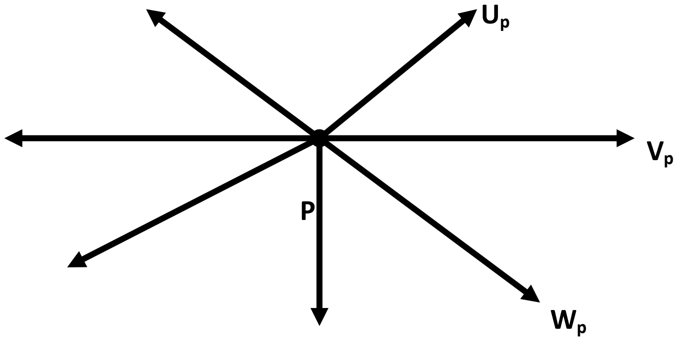

FİGURE 1.2 (TANGENT SPACE OF R^3)

THEOREM 1.1

Let (X, d) be a complete metric space and if $T: X \to X$ is a contraction, hence

i. T has one and only one fixed point which is $x \in X$

ii. For one of $x_0 \in X$ $\{T^n x_0\}$ iteration series convergens this fixed point of T. (It means for every $n \in N$, {x} iteration sequence defined by $(x_n) = T(x_{n-1})$ converges at fixed point of T. [2]

THEOREM 1.2

Let (X, d) be a complete metric space and let $T: X \to X$ be a function. If T_m is a contraction for some $m \in N$, so

i. T has if and only if one x *fixed point*.

ii. When any $x_0 \in X$ is given, (x_n) iteration sequence for every consequtive $n \in N$ defined by $(x_n) = T(x_{n-1})$ converges x fixed point of T. [2]

The first type fixed point theorem is usually practiced to prove the solution of the differential equations. In this study, the second type fixed point theorem will be practiced on the Menelaus theorem. In this theorem, we will try to find the fixed point by making use of lines creatind a triangles' borders and interceptors' director vectors of this border. Here. We will make use of a second information the homothety and dilation.

DİLATİON AND HOMOTHETY

DEFİNİTİON 2.1

When it's given a fixed "0" point and a real number k≠0, a point P' can be matched with the vectorial equation $\overrightarrow{OP'} = k\overrightarrow{OP}$.

A transformation like this is called homothety, the point "0" is a called the centre of the homothety and number k is called the homothety ratio. It's showed by $[O.k]$ symbol.
"0" is homothety center of P and the homothety ratio "k" is showed in figure 2.1.

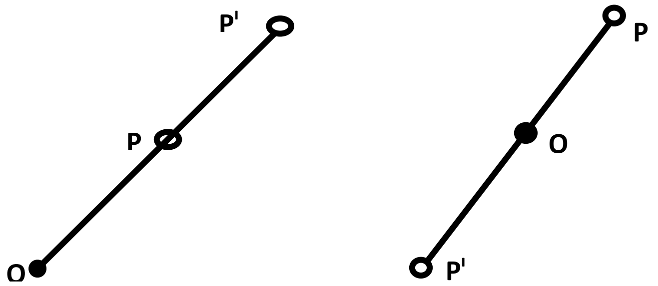

FİGURE 2.1 (HOMOTHETY)

DEFİNİTİON 2.2

A dilation is a transformation of the plane, D_k, such the 0 is a fixed point, k is a nonzero real number, and P' is the image of point P, then 0, P and P' are collinear and $\frac{OP'}{OP} = k$. This homothety showed $(PP', O) = k$ or $P = [O, k]P'$

THEOREM 2.1

The product of two homotheties which have different centres is a homothety .
$[B, \mu][A, \lambda] = [c, \lambda\mu], (\lambda\mu \neq 1)$ and A, B, C homothety centres are linear. [3].
Proof. Let's take a U point that's not on $\overleftrightarrow{AB}$ and write

$$[A, \lambda]U = U' \ and \ [B, \mu]U' = U'' \quad \text{for } U.$$ (See Figure 2.2)

Let $\overleftrightarrow{UU''}$ intercept $\overleftrightarrow{AB}$ at point C. ($\lambda.\mu$≠1there is an internal interception.)

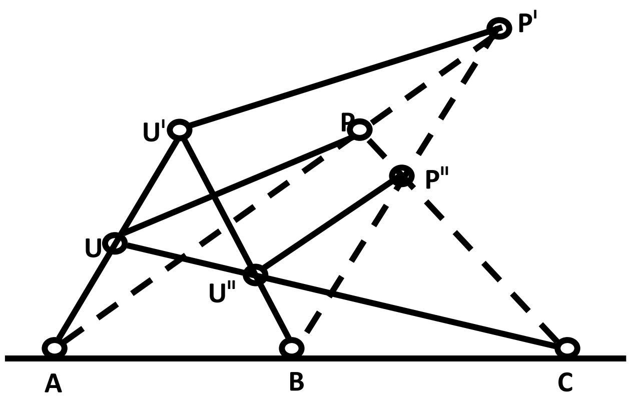

FİGURE 2.2 (THEOREM 2.1)

We want to show that the point P is different from U.
We can write [A, λ] P=P' and [B, μ]P'=P'' ,
From the internal interception,

$$\overrightarrow{U''P''} = \mu\overrightarrow{U'P'} \quad \text{and} \quad \overrightarrow{U'P'} = \lambda\overrightarrow{UP}$$

Therefore ,

$$\overrightarrow{U''P''} = \lambda\mu\overrightarrow{UP}$$

and also line $\overrightarrow{PP''}$ goes from point C and as a result [C, λμ] turns point P into point P''. [**3**]

THEOREM 2.2

If D_1 and D_2 are two dilations with magnification factors α_1 *and* α_2 respectively, then $\alpha_1 . \alpha_2$ is the magnification factors of D_2D_1. That is the magnification factor of the product of two dilations is the product of the the magnification factors.[1]
That is ,

$$d(D_2D_1(P), D_2D_1(Q)) = \propto_2 d(D_1(P), D_1(Q))$$
$$= \propto_{1.} \propto_2 d(P,Q)$$

THEOREM 2.3

A homothety, in the plane, converts a line segment into a line segment that is paralel to it.(See Figure 2.3) [3].

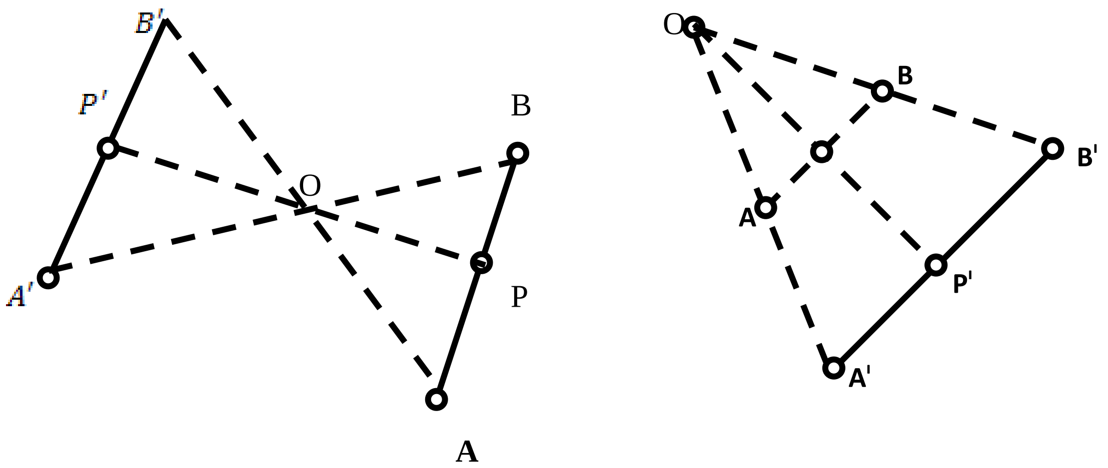

FİGURE 2.3 (THEOREM 2.3)

FİXED POİNT İN MENELAUS

THEOREM 3.1

X, Y, Z are points taken on the borders or extensions of ABC triangle and defined by the ratios $(BC, X) = \lambda$, $(CA, Y) = \mu$, $(AB, Z) = \vartheta$, the points different from the corners are linear $X, Y, Z \Leftrightarrow \lambda\mu\vartheta = +1$. (See Figure 3.1)
Proof.

$$XB/XC = \lambda \Leftrightarrow XB = \lambda XC \Leftrightarrow C = [X, \lambda]\, B$$
$$YC/YA = \mu \Leftrightarrow YC = \mu YA \Leftrightarrow A = [Y.\mu] . C$$

$$\frac{ZA}{ZB} = \vartheta \Leftrightarrow ZA = \vartheta\, ZB \quad \Leftrightarrow B = [Z, \vartheta\,].A$$

then

$$C = [X, \lambda]B = [X, \lambda].[Z, \vartheta]A = [Y', \lambda\vartheta]A$$

thus X, Y', Z *linear*.

$$A = [Y, \mu]\, C \Rightarrow C = [Y, \mu]^{-1}.A \Rightarrow$$

$$C = [Y, \mu^{-1}]\, A$$

Thus, we get the homotheties [Y, $\lambda\vartheta$] and [Y, μ^{-1}] that turn C into A. The equality of ratios is required the equality of centres vice versa. Therefore

$$\lambda\mu\vartheta = 1 \Leftrightarrow \lambda\vartheta = \mu^{-1}$$

$\Leftrightarrow$ The equality of ratios in the homotheties

$\Leftrightarrow$ The equality of centres in the homotheties $\Leftrightarrow$ Y=Y'

$\Leftrightarrow Y = Y' \in \overleftrightarrow{XZ}$ so X, Y, Z are linear.

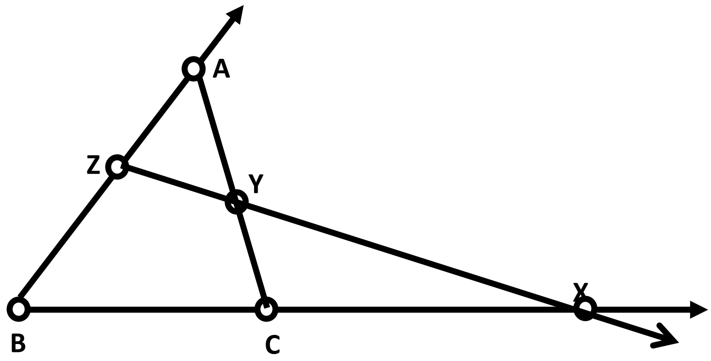

FİGURE 3.1 (*Menelaus*)

THEOREM 3.2

ABC is a triangle, let d intercept $[AB], [AC]$ *and* $[BC]$ respectively at points Z, Y and X. In this condition, for the interceptor d, C is the fixed point of ABC according to Menelaus theorem.(See Figure 3.2)

Proof P is the starting point (or point of origin) of the tangent vektor space. Let's define the transformation

$T(\vec{V}) = \vec{V} + \propto \vec{d}$ for $\forall\, \vec{d}, \vec{V} \in T_p(E^n)$ and for $\propto \in R$, $(n \in N^+)\; n \geq 2$ (See Figure 3.2).

For vector

$$\overrightarrow{PC} + \overrightarrow{CX} = \overrightarrow{PX}, T(C) = C + \propto \vec{d}, = X$$

$$T_1(C) = C + \propto_1 \vec{d}_1 = X$$

$$T_1(X) = X + \propto_2 \vec{d}_2 = B$$

$$T_2(B) = B + \propto_3 \vec{d}_2 = \vec{Z}$$

$$T_2(Z) = Z + \propto_4 \vec{d}_2 = \vec{A} \qquad (1)$$

Due to the fact that Z, Y and X are linear

$$\frac{\propto_1}{\propto_2} \cdot \frac{\propto_3}{\propto_4} \cdot \frac{|AY|}{|YC|} = 1$$

provides the Menelaus theorem.

$|AY| = \lambda_5$ and $|YC| = \lambda_6$

$$T_3(A) = A + \lambda_5 \overrightarrow{d_3} = Y$$

then $T_3(Y) = Y + \lambda_6 \overrightarrow{d_3} = C$ (2)

From (1) *and* (2) equations

If it is taken as

$$T(x) = \left(T_3\left(T_3\left(T_2\left(T_2\left(T_1\left(T_1(X)\right)\right)\right)\right)\right)\right)$$

then $Tx = x$.

As a requirement of definition 1.1, we can get that the transformation T has one fixed point creates the Menelaus theorem defined in the space of tangent vector for all iterations of the transformaton T, it is $T^n C = C$. If those algorithims practiced for point Z, Z would be the fixed point.

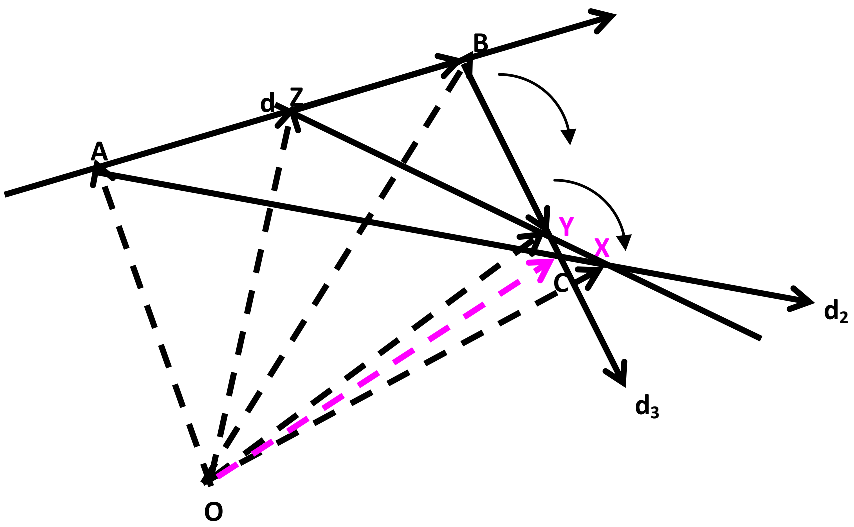

FİGURE 3.2 (THEOREM3.2)

REFERENCES

CLEMENS,S.R.1973, Fixed point theorems in Euclidean Geometry, Mathematics Teacher, April 1973,324-330.

SOYKAN, Y. 2008. Fonksiyonel Analiz. Nobel Yayın Dağıtım Ankara. Turkey.

LACHLAN,S.M.A.1893 *An Elementary Treatise on Modern Pur Geometry*, Macmillan and Co. New York.

O'NEİLL, B. 1997. *Elementary Differential Geometry*, Second Edition, Academic Press, San Diego,USA